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A STUDY OF THE APPLICABILITY OF GENERALIZED DISTRIBUTION NETWORK HEAD LOSS CHARACTERISTICS. PART II

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June 1965

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A STUDY OF THE APPLICABILITY OF GENERALIZED DISTRIBUTION NETWORK HEAD LOSS CHARACTERISTICS

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A Report on Part II of an Investigation
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Division of Water Supply and Pollution Control
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Research Grant WP-526

Conducted by
The Department of Civil Engineering
University of Illinois
Urbana, Illinois

June, 1965

PREFACE

The studies reported herein comprise Part II of an investigation on "criteria for analysis of water distribution systems". The entire investigation has been supported by the Division of Water Supply and Pollution Control, U.S. Public Health Service, under Research Grant WP-526. Other completed parts of the investigation are:

Part I - "A Study of Distribution System Equalizing Storage Hydraulics," Hydraulic Engineering Series No. 6, March, 1965.

Part III - "A Study of Power Consumption for Equalizing Storage Operating Options," Hydraulic Engineering Series No. 8, June, 1965.

Part IV - "An Analysis of Distribution Demand Variability," Hydraulic Engineering Series No. 9, June, 1965.

Work is under way on Part V, Ground-Storage Booster Pumping Hydraulics, the last part of the investigation.

Professor M. B. McPherson is the principal investigator.

ABSTRACT

If an exact solution could be obtained for the network head loss between a single input and a single equalizing storage site it would be too involved for general application in design or operation simulation. Limitations and applicability of empirical approximations are explored and limits of parameters defined.

ACKNOWLEDGEMENTS

The private communication from Dr. M. H. Diskin, appearing in Appendix II of this report, is gratefully acknowledged. The first author is indebted to Professor S. J. Fenves for his valuable suggestions on programming a network node balancing solution.

TABLE OF CONTENTS

	Page
INTRODUCTION	1
DEFINITIONS OF SYMBOLS	3
STUDY LIMITATIONS	5
EQUATIONS OF STATE	6
Figure 1 - Notation Example	7
SOME PROPERTIES OF SIMPLE SYMMETRICAL NETWORKS	11
Figure 2 - Simple Symmetrical Network	11
Figure 3 - Solution of a One-Loop Symmetrical Network	14
Figure 4 - Solution of a One-Loop Symmetrical Network (Log-Log)	15
Figure 5 - Symmetrical Network with Two Loops	13
Figure 6 - Solutions for the Network in Figure 5	17
APPROXIMATION METHODS	19
Table 1 - Example of Two-Point Fit, Data	21
Figure 7 - Example of Two-Point Fit, Plot	22
Figure 8 - Example of a Poor Two-Point Approximation	23
MORE COMPLEX NETWORKS - DESCRIPTION	24
Figure 9 - Topological Pattern of Branches and Junctions	25
Figure 10 - Pipe Resistance Patterns	26
Figure 11 - Demand Load Patterns	27
Figure 12 - Position of the Sixth Storage Position	28
EFFECT OF STORAGE LOCATION	29
Figure 13 - Three Relative Positions of the Storage Reservoir	30
Figure 14 - Schematic Representation of the Network in Figure 2	31
Figure 15 - Network of Figure 2; Φ = constant, pK = constant, p varied	33
Table 2 - Changes in Φ and N for the 36-U1-U1-S Series Networks, with the Storage Position Varied	35
EFFECTS OF LOADING AND RESISTANCE PATTERNS	36
Table 3 - Changes in Φ and N with Demand Load Pattern Variation	36
Table 4 - Effect on N and Φ with Variation of the Resistance Pattern	37

TABLE OF CONTENTS (Continued)

	Page
EFFECT OF SIZE OF NETWORK.	38
Table 5 - Effect of Varying Network Size.	38
Figure 16 - Effect of Variation in Network Size	39
CONCLUSIONS.	40
REFERENCES	42

APPENDIX	CAPTION	
I	A DEMONSTRATION OF SOLUTION UNIQUENESS IN NETWORK ANALYSIS, by R. A. Wiseman	I-1 to I-9
II	A STUDY OF ALTERNATIVE EXPRESSIONS DESCRIBING GENERALIZED DISTRIBUTION NETWORK HEAD LOSS CHARACTERISTICS, by MacKenzie L. Davis	II-1 to II-14
III	TABLES LISTING RESULTS FOR ALL COMPUTER RUNS	III-1 to III-11

INTRODUCTION

This study is concerned with water distribution systems having a single supply, pumped or by gravity from a ground reservoir, and a single equalizing storage site. Such a system is admittedly a rather simple case, but results obtained should provide inferential guides for appraising more complex systems with multiple sendouts and/or storage sites.

In order to simulate system performance, a continuous system balance must be made over the full period of operation simulation. A system balance is merely a reconciliation of flows and heads for a given system over a selected time interval, such as an hour, in sequential order of demands[1-3]*. A given hour of demand with its associated sendout rate and flow rate into or out of storage usually comprise a unique combination, for which network head loss magnitudes and distributions are also unique. For a given demand loading the head loss across a network is a function of sendout and storage rates which in turn are affected by sendout and storage heads. In a complete analog system simulation this feedback would be accommodated directly, but a satisfactory machine does not exist (1965). A digital computer can be utilized to converge on balanced system conditions in a series of trial iterations. However, each iteration would necessarily include a separate network balance. The high cost of the vast number of trial network balances involved in an iterative, direct system simulation has precluded its application.

An effort was made to synthesize characteristics of network head losses in a simple equation form so that the number of network balances required for system balances could be reduced to only two or three for determination of

*: References are cited at the end of the text.

equation constants[1]. It was deduced that the head loss between the sendout and equalizing storage, Σh, could be expressed as

$$\Sigma h/Q_d^m = f\ (Q_p/Q_d) \qquad (1),$$

where Q_d is the total customer demand drawn from the network and Q_p is the sendout rate. The exponent m is the same as that in the expression $h = f\ (Q^m)$ for the head loss in an individual network branch. The functional relationship of Eq. (1) was assumed to be satisfied by a coefficient and an exponent

$$\Sigma h/Q_d^m = \phi_o\ (Q_p/Q_d)^N \qquad (2),$$

and it was shown that Eq. (2) was a satisfactory replication for four arterial network cases. These cases were simulations of actual arterial networks and network head losses had been obtained in conjunction with design studies using a McIlroy Fluid Network Analyzer[4].

Equation (2) is restricted to proportional loading conditions. For proportional loading, each customer demand load is assumed to vary in direct proportion with the total network customer demand. While fire flows and heavy industrial demands are not proportional, domestic loads for fairly homogeneous service districts tend to be proportional.

The objectives of this study included: (1), determination of a function describing network head loss characteristics which would be superior to Eq. (2) with regard to fidelity of representation and logical formulation; (2), evaluation of Eq. (2) from the standpoint of fidelity of representation; and (3), identification of network configuration effects on the exponent N of Eq. (2). These objectives are of vital concern because Eq. (2) was incorporated in the system relations used to determine criteria for determination of equalizing storage requirements[3] and in a series of operation simulations[5].

DEFINITIONS OF SYMBOLS

$[A]$ = incidence matrix, branches and nodes.

$[B]$ = incidence matrix, branches and loops.

a = connectivity element indicating the relationship between a branch and a junction.

b = connectivity element indicating the relationship between a branch and a loop.

D = demand load at a junction.

h = head loss in a branch

Q = flow in a branch.

S = a direction function indicating the direction of flow in a particular branch in relation to its initialized direction.

k = resistance of a branch, a pipe coefficient accounting for friction factor, length, diameter and units, Eq. (3).

k' = "base branch" resistance, the smallest network branch resistance.

m = exponent in Eqs. (1), (2) and (3).

N = exponent in Eq. (2).

ϕ_o = constant in Eq. (2), (length/(flow rate)^2).

ϕ = nondimensional constant in Eq. (16).

Σh = head loss between input junction and storage junction.

Q_p = input flow rate, (volume/unit time).

Q_d = total network demand; $Q_d = \sum_{i=1}^{n} D_i$; (volume/unit time).

Q_s = flow rate to or from equalizing storage; $Q_s = Q_p - Q_d$; (volume/unit time).

ΣB = number of branches between the input junction and the storage junction, taken over any direct route.

K = $k' \Sigma B$

K_o = sum of k values for branches between the input junction and the storage junction (simple, symmetrical networks, only).

$p = k/K_o$

nb = number of branches in a network.

nj = number of junctions in a network.

nL = number of loops in a network.

Subscripts:

i or i' denotes a joint (or node);

j denotes a branch;

k denotes a loop.

STUDY LIMITATIONS

The networks analysed are idealized models which are intended to approximate the effects in an actual water distribution system arterial network. Simplifying assumptions normally used in design are inherent in the formulation of the idealized models.

It is assumed only arterial mains are to be considered and all demand loads at grid connections to arterial mains are consolidated and relocated at nearby arterial main junction points. Proportional loading is adopted. Local head losses at junction points are assumed accounted for in the resistivity of adjacent branches. The head loss in a branch between two junction points is assumed to be governed by a conductivity relation having a fixed exponent

$$h = S_j \, k_j \, Q_j^{\,m} \qquad (3),$$

where

$$S = Q/|Q| = h/|h|$$

indicates the direction of flow.

A fixed m of 1.85 is specified in the empirical Hazen-Williams equation. The range of m is 1.75 to 2 for turbulent pipe flow, the limits being for hydraulically smooth and hydraulically rough performance, respectively. An m of 2 has been used throughout this report primarily for simplification of mathematical developments and computations. It is assumed that other values of m would give comparatively similar results. However, there are good arguments for using an m of 2 in practical network analyses[2]. In network balances, use of only one value of m is permissible.

EQUATIONS OF STATE

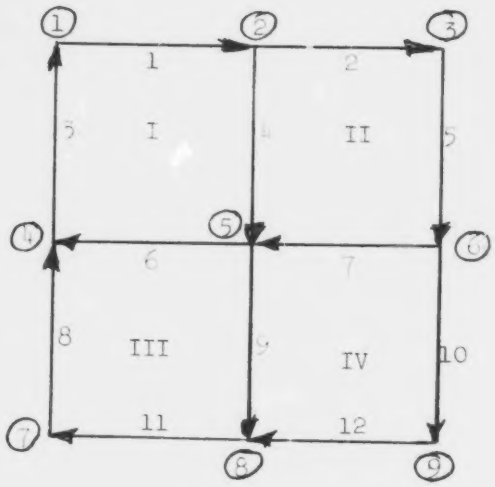
The development of a flow solution for the idealized water distribution network with a given set of boundary conditions requires simultaneously satisfying the conductivity equation, Eq. (3), and the two governing requirements for a connected network: conservation, or continuity, of flow at every junction and a single-valued head at every junction. The first of these two requirements is satisfied by a set of equations generally called junction equations, or node equations,

$$\sum_{j=1}^{nb} a_{ij} Q_j - D_i = 0 \quad \text{for } i = 1,2,\ldots.(nj - 1) \qquad (4).$$

Junctions are denoted by the subscript i, branches by the subscript j and independent loops by k. Each are numbered consecutively in an arbitrary fashion, such as in Figure 1. The number of junctions is given by nj, the number of branches by nb, and the number of loops by nL. In addition to numbering, each branch is given an initialized arbitrary flow direction.

FIGURE 1

NOTATION EXAMPLE



Example of Numbering System

(1) - Nodes

1 - Branches

I - Loops

(nj = 9 junctions,

nb = 12 branches,

nL = 4 loops).

Incidence Matrix [A]

Nodes i \ Branches j	1	2	3	4	5	6	7	8	9	10	11	12
1	1	0	-1	0	0	0	0	0	0	0	0	0
2	-1	1	0	1	0	0	0	0	0	0	0	0
3	0	-1	0	0	1	0	0	0	0	0	0	0
4	0	0	1	0	0	-1	0	-1	0	0	0	0
5	0	0	0	-1	0	1	-1	0	1	0	0	0
6	0	0	0	0	-1	0	1	0	0	1	0	0
7	0	0	0	0	0	0	0	1	0	0	-1	0
8	0	0	0	0	0	0	0	0	-1	0	1	-1
9	0	0	0	0	0	0	0	0	0	-1	0	1

Incidence Matrix [B]

Loops k \ Branches j	1	2	3	4	5	6	7	8	9	10	11	12
1	1	0	1	1	0	1	0	0	0	0	0	0
2	0	1	0	-1	1	0	1	0	0	0	0	0
3	0	0	0	0	0	-1	0	1	1	0	1	0
4	0	0	0	0	0	0	-1	0	-1	1	0	1

The element a_{ij} indicates the association between the jth branch and the ith junction. As in the example matrices of Figure 1, if the jth branch connects to the ith node a non-zero value of either +1 or -1 is assigned to the element; the respective sign depends upon whether the initialized arbitrary flow direction of the jth branch points away from the node or towards it. A branch is assigned a value of zero if it does not connect to the ith node. The grouping of all the a_{ij} terms of the network into an orderly matrix gives the network incidence matrix, $[A]$. Two basic characteristics of $[A]$ may be noted: (1), each column has exactly two non-zero values, one positive and one negative; (2), the number of non-zero elements in each row is equal to the number of branches connecting the particular node. The first characteristic occurs because, by definition, each branch terminates at its adjacent two nodes.

In essence, Eq. (4) indicates that the summation of flow from all sources into and out of a junction is equal to zero, i.e., there must be no accumulation of fluid at the junction. Because there is no accumulation at or between junctions, the summation of outflow (the total network demand plus any flow going to storage) must be equal to the input*. Consequently, when the requirement for continuity of flow at a junction is applied to all nj junctions, there will be one redundant equation. For this reason Eq. (4) is applied to only the first (nj-1) equations.

The connected network requirement of a single-valued head at every junction leads to the so-called "loop" equations. For a summation of head drops (or rises) in a traverse around a closed loop, upon returning to the starting point the head there must be the same as it was initially, or

$$\sum_{j=1}^{nb} b_{kj}\, h_j = 0 \qquad \text{for} \quad k = 1,2,\ldots nL \qquad (5).$$

The terms b_{kj} have a function similar to the term a_{ij} in Eq. (4), and are elements of the connectivity matrix $[B]$, indicating the association of the jth branch with the kth loop. If the jth branch is in the kth loop, a non-zero value of either +1

*: (See Appendix I for a more formal presentation).

or -1 is assigned to the element, the sign depending respectively on whether the initialized direction of loop flow is in the same direction as the assigned branch direction or is in the opposite direction. A zero is assigned to the element if the jth branch is not in the kth loop.

Basically, there are two approaches for solving water distribution network Eqs. (3), (4) and (5). The approach used would be determined by which one of two sets of variables, junction heads or branch flows, were chosen as independent variables. The ultimate choice might depend upon the boundary conditions, but often either method could be readily applied. However, the branch flow approach has been the overwhelmingly predominant choice.

If one desires to derive a set of equations that incorporate the branch flows Q_j as the independent variables*, the form of the Eq. (5) set must

*The necessary equations can also be developed using junction heads, H, as the variables. However, because the junctions are of primary concern instead of the branches, the branch parameters should be identified in terms of adjacent junctions. Therefore, the subscript j denoting the jth branch is replaced by a pair of subscripts i and i' which denote the two junctions connected by the jth branch (for example, $Q_j = Q_{ii'}$). Eq. (3) is now rearranged so that the branch flow is the dependent variable:

$$Q_{ii'} = \frac{S_{ii'}}{(k_{ii'})^{1/m}} (H_i - H_{i'})^{1/m} \qquad (3b),$$

where the term $S_{ii'}$, which is equal to S_j or $1/S_j$, has a special relationship in the conductivity equation above in that it is not affected by the exponent manipulations. Changing the subscripts in Eq. (4) and rearranging:

$$\sum_{\substack{i'=1 \\ i' \neq i}}^{nj} c_{ii'} Q_{ii'} - D_i = 0 \qquad \text{for } i = 1,2,\ldots.(nj-1) \qquad (4b),$$

where the elements $c_{ii'}$ are related to the elements a_{ij} by the matrix equation $[C] = [A][A^t]$. Substitution of Eq. (3b) into Eq. (4b) gives

$$\sum_{\substack{i'=1 \\ i' \neq i}}^{nj} c_{ii'} \frac{S_{ii'}}{(k_{ii'})^{1/m}} (H_i - H_{i'})^{1/m} - D_i = 0,$$

$$\text{for } i = 1,2,\ldots.(nj-1) \qquad (4c).$$

The requirement of a single valued junction head is automatically satisfied in the solution of the above equation for a particular set of boundary conditions (see Appendix I for a uniqueness proof).

be changed so that it includes the branch flows as variables instead of the branch head loss terms h_j. Substitution of Eq. (3) into the Eq. (5) set will accomplish this change, giving a more familiar form of the loop equations:

$$\sum_{j=1}^{nb} b_{kj} S_j Q_j^m = 0 \quad \text{for } k = 1,2,\ldots nL \qquad (6).$$

Because nb = nL + (nj - 1) in a "properly" connected network (no more than one branch directly connects two junctions), Eq. (4) and (6) sets have nb unknowns and nb independent equations of which nL of these equations will have nonlinear elements.

Although theoretically possible, it is highly impracticable to solve simultaneously Eqs. (4) and (5) sets exactly by substitution and elimination methods for m = 2; and impossible if m is not an integer. Thus, the equations are normally solved by means of iterative techniques. If an iterative technique is used, the general nature of the network equation enables one to satisfy the two sets of conditions separately. Equation (4) can be satisfied by assigning initially assumed values to the branch flows Q_j such that there is no imbalance of flow at any of the junctions. If the corrections are loop-flow corrections, Eq. (4) will be satisfied at all times, because the loop-flow correction procedure would both add to and subtract from a junction the same amount, with no net gain of flow at the junction.

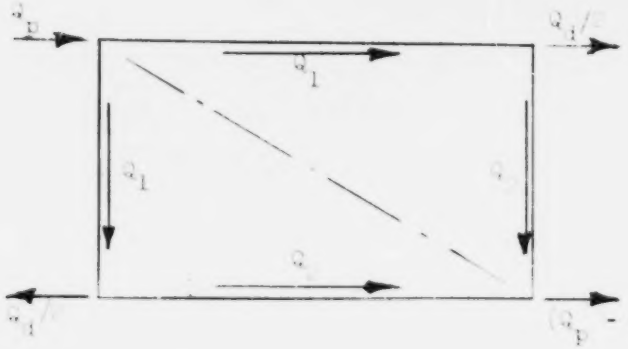
The best known iterative technique used to solve loop equations is the Hardy Cross method[6,7]. Professor Hardy Cross recognised the possibility of solving the node equations rather than the loop equations, but gave no details[6]. Except for the first two, simple examples, all network combinations to be discussed were solved by using the loop equation approach.

SOME PROPERTIES OF SIMPLE SYMMETRICAL NETWORKS

Exact solutions for two very simple networks will be discussed to give some insight into the relationships between the different parameters involved. The first example is the one-loop symmetrical network shown in Figure 2. Symmetry is about a diagonal between the input and storage nodes, with respect to branch resistances and demand loads. Consequently, the branch flows and head losses are also symmetrical about the diagonal.

FIGURE 2

SIMPLE SYMMETRICAL NETWORK



$$k_1 = p\ K_o$$

$$k_2 = (1 - p)\ K_o$$

The resistance of a first branch, connecting the input node to a demand node, is taken as a fraction p of an arbitrary total resistance K_o, the sum of either pair of resistances between the input node and the storage node. The resistance of either second branch is then $(1-p)\ K_o$.

Although there are four branch flows in Figure 2 only two need be determined because the flow is symmetrical about the diagonal. The sole loop equation for this network is satisfied exactly by the initial condition of symmetry but its use is not necessary for solving the two branch flows. The two branch flows can be determined directly from the junction equations alone and are linear functions of the boundary conditions Q_p and Q_d:

$$Q_1 = Q_p/2 \tag{7},$$

$$Q_2 = (Q_p - Q_d)/2 \tag{8}.$$

The head loss, Σh, across the network from the input node to the storage node is the sum of either pair of branch head losses between these points. Using Eqs. (3), (7) and (8), head loss across the network is

$$\Sigma h = K_o \left\{ p \frac{Q_p^2}{4} + S_2 (1 - p) \left(\frac{Q_p - Q_d}{4} \right)^2 \right\} \tag{9}.$$

Rearranged into a nondimensional form, Eq. (9) becomes

$$\frac{\Sigma h}{K_o Q_d^2} = \frac{1}{4} \left\{ p \left(\frac{Q_p}{Q_d} \right)^2 + S_2 (1 - p) \left[\left(\frac{Q_p}{Q_d} \right)^2 - 2 \frac{Q_p}{Q_d} + 1 \right] \right\} \tag{10}$$

where $$S_2 = \frac{(Q_p/Q_d - 1)}{|Q_p/Q_d - 1|} .$$

To illustrate the nature of Eq. (10), graphically, three different values of p are employed. For each p, values of $\Sigma h/K_o Q_d^2$ are plotted versus Q_p/Q_d in Figure 3. For $p = 1/2$ the flow resistances of all four pipes are equal. The other two p-values represent limiting cases. For $p = 0$ the total demand load is concentrated at the input node, and for $p = 1$ the total demand load is concentrated at the storage node.

The most significant characteristic of Eq. (10) is that its general form is of the type given by

$$\Sigma h/K_o Q_d^2 = \alpha \left(\frac{Q_p}{Q_d} \right)^2 + \beta \left(\frac{Q_p}{Q_d} \right) + \gamma \tag{11}.$$

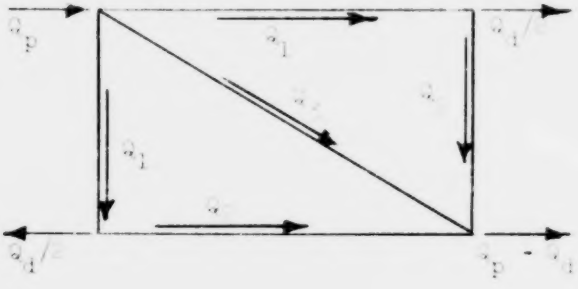
However, α, β and γ are not constant over the entire range of Q_p/Q_d but are discontinuous at some particular Q_p/Q_d because all three coefficients are a function of the direction function S_2. Thus, the whole equation is discontinuous (more correctly, one of its derivatives is discontinuous) at this critical value of Q_p/Q_d. Although there is only one such critical point in the above case, the number of discontinuous points could become quite large for a large network. The number of critical points depends upon the number of network branches in which the flow can become zero for some value of Q_p/Q_d.

Use of the direction function S_j becomes quite awkward when one attempts to solve exactly a network whose flows are nonlinear functions of the boundary conditions, Q_p and Q_d. In fact, one cannot obtain an explicit expression for the branch flow terms Q_j algebraically because of their absolute value representation in the function S_j. To circumvent using the function S_j, one must determine its sign beforehand by determining the critical values of Q_p/Q_d and the direction of the branch flows in the ranges between the critical values. This particular difficulty is encountered in the next example.

The network in Figure 5 differs from the symmetrical one-loop network of Figure 2 in that it has a fifth branch along the diagonal of symmetry, which divides the network into two identical loops. Only the specific case in which all branches are of equal resistances is considered.

FIGURE 5

SYMMETRICAL NETWORK WITH TWO LOOPS



$k_1=k_2=k_3=k$

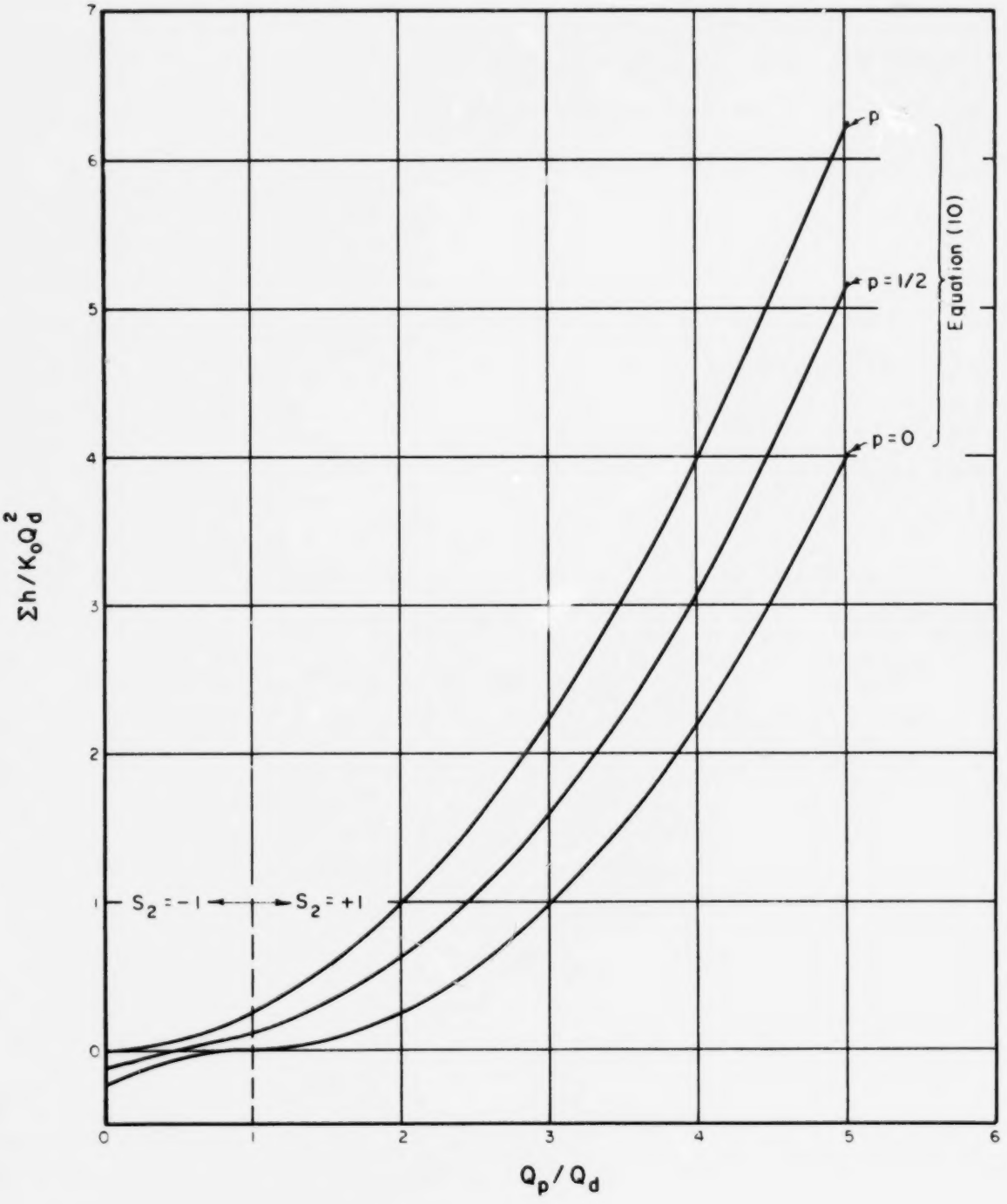


FIG. 3 SOLUTION OF A ONE-LOOP SYMMETRICAL NETWORK

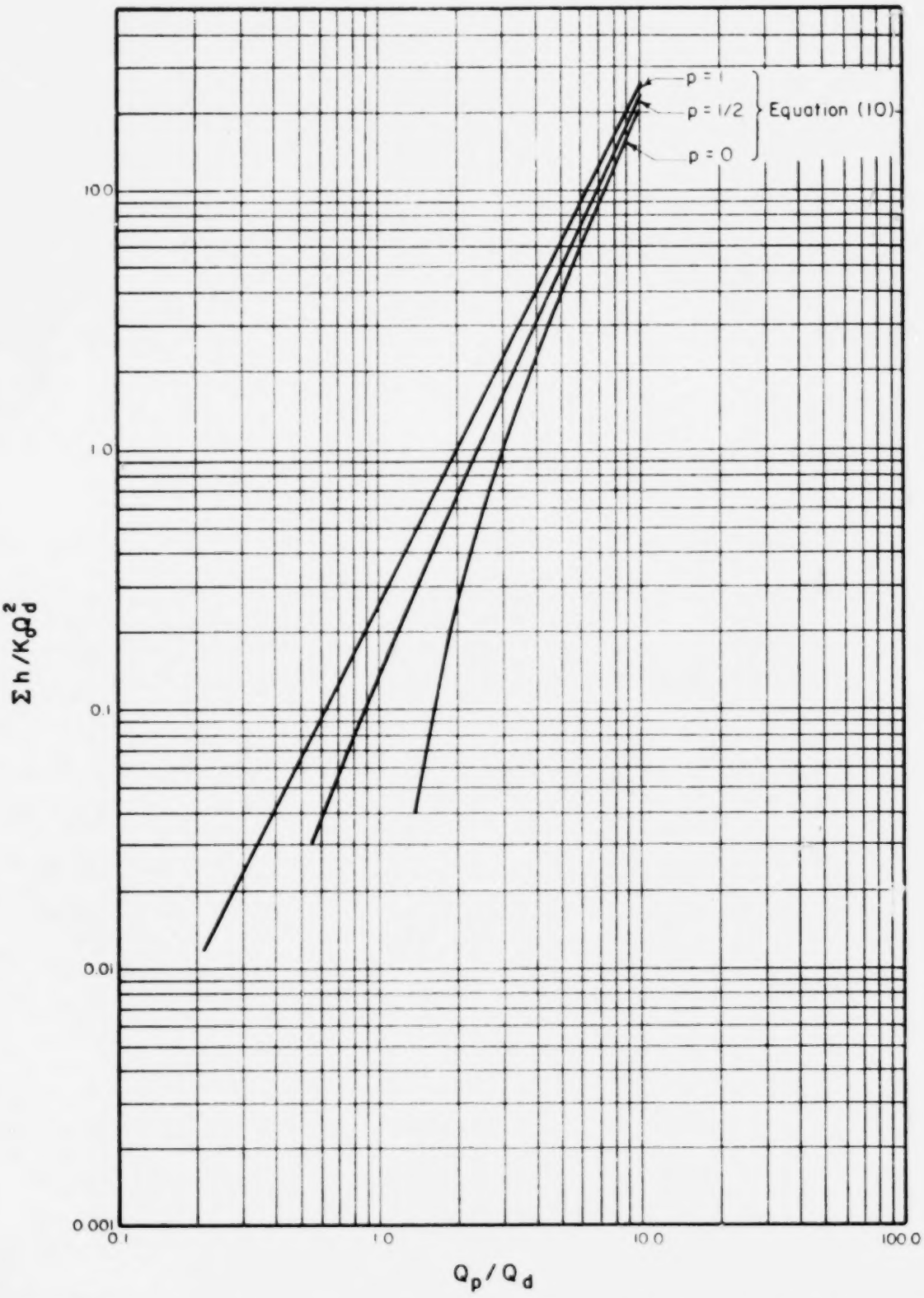


FIG. 4 SOLUTION OF A ONE-LOOP SYMMETRICAL NETWORK

As in the first example, solution of only two simultaneous equations is necessary because of symmetry. Of the three remaining equations of equilibrium for the two-loop network, one of them is a nonlinear loop equation, Eq. (12), and the other two are junction equations, Eqs. (13) and (14). In the presence of a nonlinear loop equation, the network is called a nonlinear network.

$$S_1 k' Q_1^2 + S_2 k' Q_2^2 - S_3 k' Q_3^2 = 0 \qquad (12).$$

$$2Q_1 + Q_3 - Q_p = 0 \qquad (13).$$

$$2Q_2 + Q_3 - Q_p + Q_d = 0 \qquad (14).$$

As was stated in the preceding discussion, use of the direction function S_j becomes quite awkward when nonlinear loop equations are involved. To preclude the need for using S_j, the flow problem must be separated into three ranges of Q_p/Q_d. The critical values of Q_p/Q_d for the network in this example are 0.50 and 1.50. The exact solutions along with the directions of the branch flows for the three ranges are shown in Fig. 6.

FIGURE 6

SOLUTIONS FOR THE NETWORK IN FIGURE 5

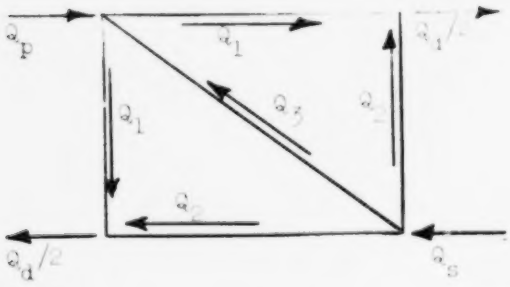


Fig. 6a

$$\frac{Q_p}{Q_d} \leq 0.50$$

$$\Sigma h/K_o\, Q_d^2 = -\frac{1}{8}\left\{3.0 - 4\left(\frac{Q_p}{Q_d}\right) - \sqrt{5.0 - 8\left(\frac{Q_p}{Q_d}\right)}\right\} \qquad (15a).$$

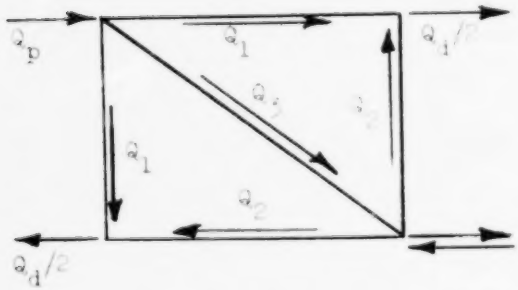


Fig. 6b

$$0.50 \leq \frac{Q_p}{Q_d} \leq 1.50$$

$$\Sigma h/K_o\, Q_d^2 = \frac{1}{8}\left\{4\left(\frac{Q_p}{Q_d}\right) - 1.0 - \sqrt{8\left(\frac{Q_p}{Q_d}\right) - 3.0}\right\} \qquad (15b).$$

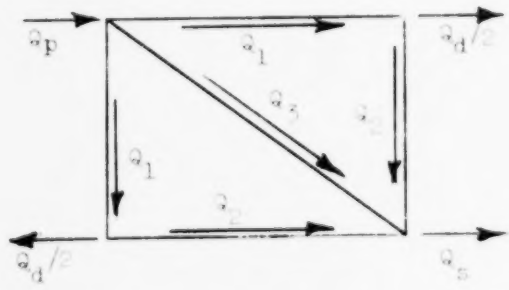


Fig. 6c

$$\frac{Q_p}{Q_d} \geq 1.50$$

$$\Sigma h/K_o\, Q_d^2 = \left\{3\left(\frac{Q_p}{Q_d}\right)^2 - 3\left(\frac{Q_p}{Q_d}\right) + 1 - \frac{1}{2}\left(2\left(\frac{Q_p}{Q_d}\right) - 1\right)\sqrt{8\left(\frac{Q_p}{Q_d}\right)^2 - 8\left(\frac{Q_p}{Q_d}\right) + 3}\right. \qquad (15c).$$

Two conclusions can be drawn from the preceding example. First, due to the presence of a nonlinear governing equation, the solutions of the nonlinear network in Figure 6 given by Eqs. (15) are not of the form given by Eq. (11); instead, Eqs. (15) have an additional term which is a square root function. If one attempted to develop a general equation such as Eq. (11) for simple, multiple-loop networks, an undesirably large number of coefficients would be involved. The second conclusion is that the development of an exact solution would be extremely tenuous for two reasons: (1), the change in branch flow direction as Q_p/Q_d varies; and (2), the complexity involved in eliminating terms in a nonlinear equation.

APPROXIMATION METHODS

As stated in the introduction, one of the objectives of this study was to develop a mathematical model that would serve as an approximate equivalent to idealized pipe networks. The preceeding examples indicate the difficulty involved in finding an exact solution. It is evident that there is no obvious, simple series approximation, although a relatively long series made up of powers of Q_p/Q_d would probably be satisfactory, with the exactness of replication depending on the number of terms considered. Before deciding on any particular method for approximating a network solution, the relative importance of some of the governing requirements should be considered.

Some of the principal requirements that need to be considered are: the level of precision desired in approximating an exact solution, the number of network balances under different loading conditions necessary to define the parameters of the equation, and the relative simplicity of the approximating equation for use in system simulation studies. It is essential that all of these requirements be met to some degree. If a very precise approximation to an exact solution is desired, a rather complicated expression would be needed; however, the other two requirements would then not be met. Thus, some sort of compromise is necessary.

The suitability of Eq. (2) for approximating h was extensively investigated using a comprehensive series of idealized networks. However, a nondimensional modification of Eq. (2) was adopted so that the units of i_o, h and Q_d are immaterial in evaluating the results:

$$h/K Q_d^2 = \sum (Q_p/Q_d)^N \qquad (16)$$

Both sides of Eq. (2) are divided by the parameter K, with $\Phi = \Phi_o/K$. The term K is the number of individual pipes taken over any direct route between the input point and the storage point, ΣB, multiplied by the base branch resistance, k'. The base branch resistance is the smallest branch resistance of the network. Although Eq. (16) is a simple type, there are very definite limits on its suitability in approximating a network flow solution.

Except for some limiting cases, it is possible to have negative values for the head loss parameter Σh. This situation arises when the head loss from the storage position to the network region of lowest head is greater than the head loss from the input to this region. In a symmetrical network with a symmetrical distribution of demand loads (for example, the network of Figure 2 with $p = \frac{1}{2}$), negative values of Σh would occur whenever the flow from the storage reservoir was greater than the flow from the input ($Q_p/Q_d < 0.50$). Obviously, Eqs. (2) and (16) are not applicable for this range of Q_p/Q_d. However, as the magnitude of Q_p/Q_d increases, the exact solution tends to be satisfied more closely by a simple power relation such as Eq. (16) (see Figure 4). In fact, as Q_p/Q_d approaches infinity, the value of the exponent N will approach 2, as would be expected. With minor reservations, Eq. (16) is a good replication for the higher ranges of Q_p/Q_d where a log-log plot of the true solution closely approximates a straight line, but will not necessarily be simultaneously representative of the lower ranges of Q_p/Q_d.

Two characteristics tend to offset the difficulties encountered when the Q_p/Q_d ratio is near the critical value. The head loss across the network Σh is necessarily low, and the outflow from the storage reservoir is approximately equal to the input. The low Σh tends to lessen the effect of a large error. In practice, a tank outflow equal to the system input is rarely encountered.

In an effort to insure coverage of a reasonable practical range and to provide a consistent basis for comparison, values of Q_p/Q_d of 0.75 and 5 were used, to obtain the two parameters Φ and N for each network case. An example of the effect of this approximation and the error involved is illustrated in Table 1 and Figure 7. It must be noted that the values of Φ and N obtained are somewhat ficticious because their value depends considerably upon the range and magnitudes of Q_p/Q_d used.

TABLE 1

EXAMPLE OF TWO-POINT FIT (See Figure 7)

Q_p/Q_d	$\Sigma h/KQ_d^2$	$(Q_p/Q_d)^N$	Difference	Difference, %
0.500	0.0000	---	---	---
0.600	0.0060	0.0089	0.0029	48.4
0.750	0.0150	0.0150	0.0000	0.0
1.000	0.0306	0.0297	0.0008	2.6
1.500	0.0799	0.0778	0.0022	2.8
2.000	0.1623	0.1537	0.0086	5.6
3.000	0.4254	0.4015	0.0239	6.0
4.000	0.8203	0.7936	0.0267	3.4
5.000	1.3464	1.3464	0.0000	0.0
10.000	5.9521	6.9542	1.0021	14.4

(Network 36-U1-U1-S1)

Although only two network balances are essential to determine Φ and N for Eq. (16), a third network balance is recommended as a check to make sure that the desired degree of precision is obtained. The necessity for this precaution is exemplified in Figure 8. For this particular network situation the use of Q_p/Q_d = 0.75 and 5.00 gives a very poor approximation to the solution. An explanation is given in a subsequent section as to when this type of result might be expected.

A discussion of alternative approximation methods is given in Appendix II.

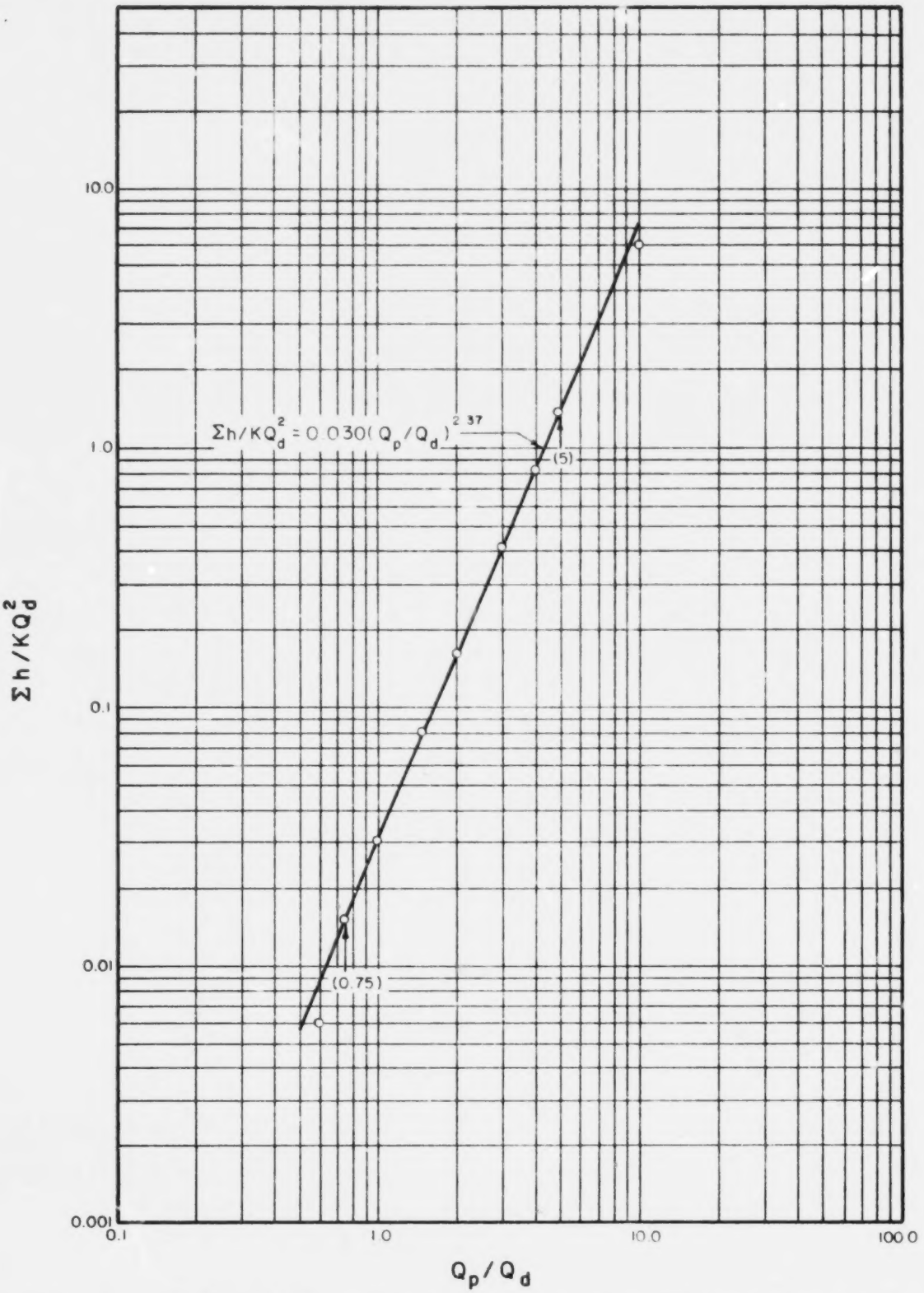


FIG. 7 EXAMPLE OF TWO-POINT FIT (NETWORK 36-UI-UI-SI)

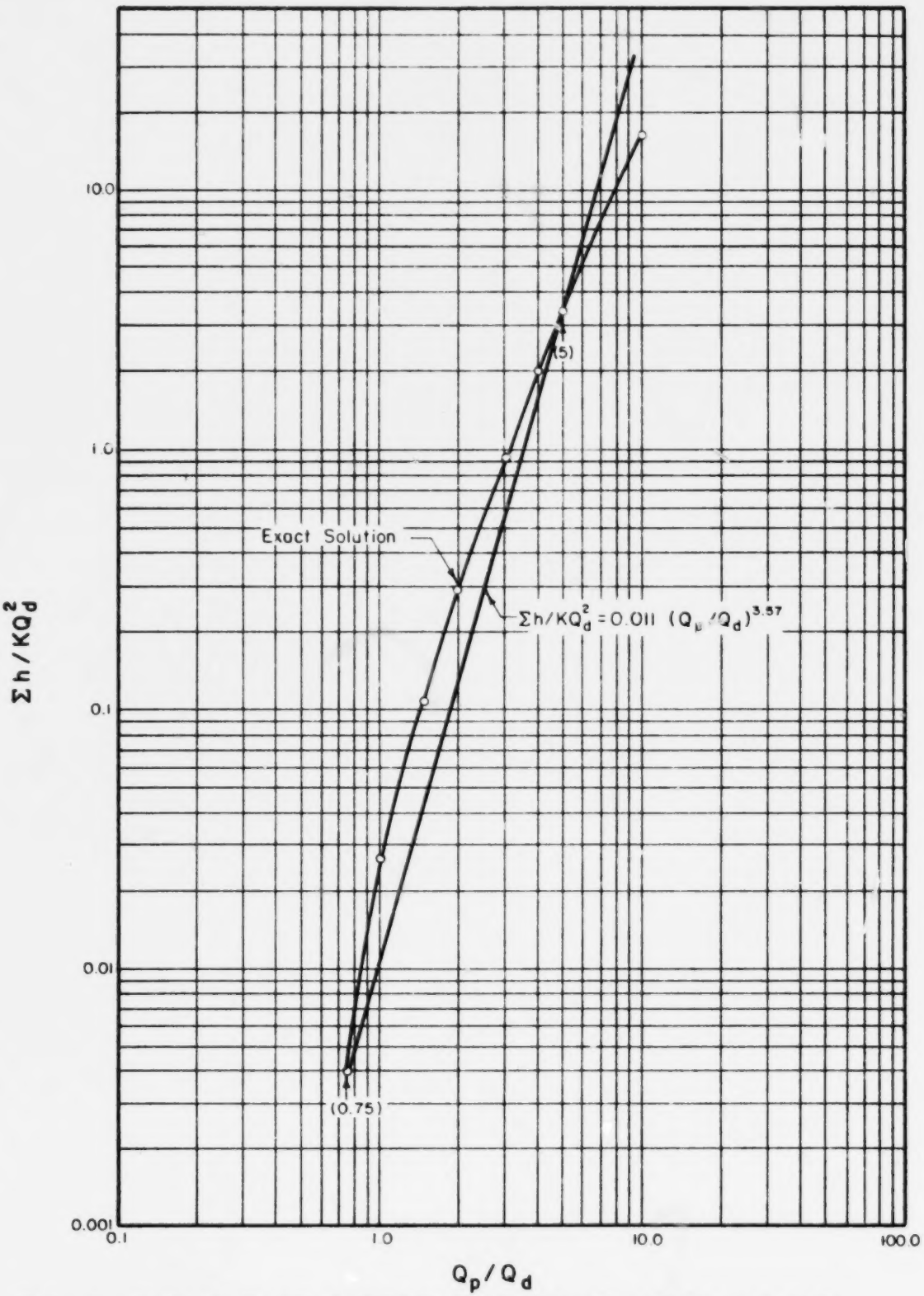


FIG. 8 EXAMPLE OF A POOR TWO-POINT APPROXIMATION
(NETWORK 36-UI-UI-S6)

MORE COMPLEX NETWORKS - DESCRIPTION

One of the main objectives of this study was to analyze the effects of varying different water distribution network characteristics and imposed boundary conditions. The most desirable procedure would be a complete separation of each characteristic and its associated effects. This separation could be accomplished only if each different characteristic, whether implicity or explicity defined, could be held essentially constant while others were varied. With large, complex networks, complete separation of characteristics and their effects would be virtually impossible. For this reason it is necessary to study variations and their effects using simple networks. Generalizations noted for simple networks can then be qualitatively applied to more complex networks.

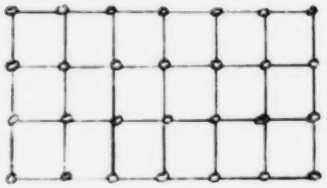
Although the number of possible variations in characteristics is almost limitless, there are in general only a limited number of basic characteristics: (1), the boundary of the network defined by the input and output points or nodes; (2), the topological pattern interconnecting these points; (3), the resistance pattern of the interconnecting branches; (4), the distribution pattern of demand loads; and (5), the position of equalizing storage in relation to the rest of the network.

Considering the boundary first, a particular pattern was used almost without exception for more complex networks so that general similarity would be preserved regardless of the number of nodes considered. The number of boundary points nj was always taken such that the square root of nj was one of the set of consecutive integers 2, 3, 4, 5, 6, or 7. Because the square root of nj was always an integer, the nodes could be arranged in an orderly square array with an equal number of nodes in all columns and rows as in Figure 9a.

TOPOLOGICAL PATTERN OF BRANCHES AND JUNCTIONS

FIGURE 9a

FIGURE 9b



The basic purpose of a water distribution network is to connect the demand load output nodes to the input and storage nodes. There are often a considerable number of possible connection paths even when the resistances of the lines connecting the nodes are disregarded. However, except for the simple networks cited earlier the only topological pattern used was the pattern illustrated in Figure 9b, where the nodes are connected by branches in a manner such that a regular, completely connected rectangular grid pattern results.

Because four basic characteristics were investiated, each with several variations, the number of combinations are considerable; and it was necessary to devise an identification code. This code is in four parts; one part for each of the four basic characteristics studied. An example would be 36-M5-U1-S1. The first part of the code indicates the size of the network by giving the number of basic loops (loops which do not have a smaller loop within themselves). The second part indicates the resistance pattern, the third part the demand load pattern and the last part the position of the storage tank.

Basically, six different resistance patterns were investigated (see Figure 10): the uniform resistance pattern U1 in which all of the branches were of the same resistance; three resistance patterns which had a large-small relative resistance grid pattern (M, N, and O); pattern L1 in which the relative resistance of a pipe depended upon which one of three different zones of the network it was in, with the highest resistance zone located across the middle of the network,

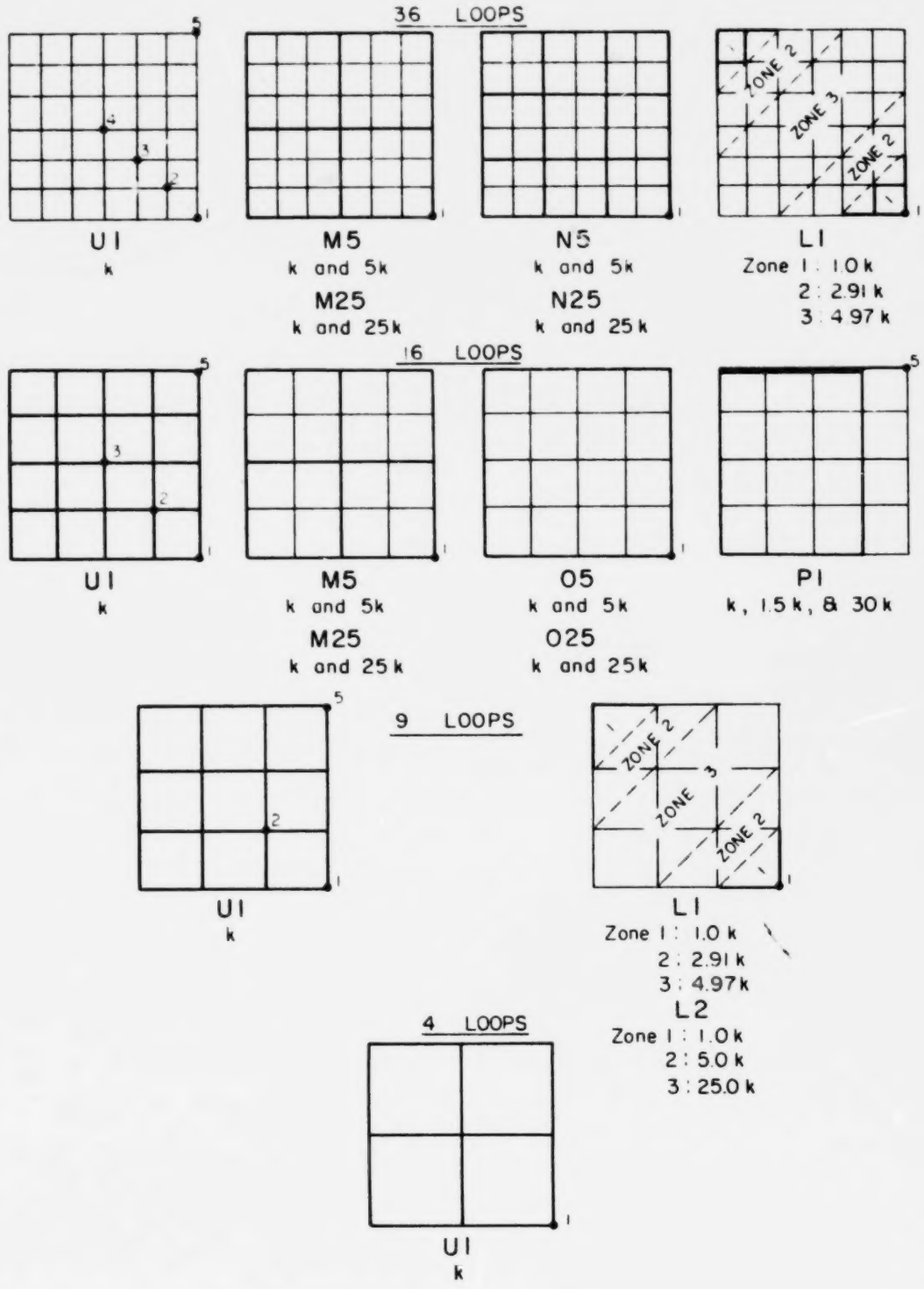


FIG. 10 PIPE RESISTANCE PATTERNS

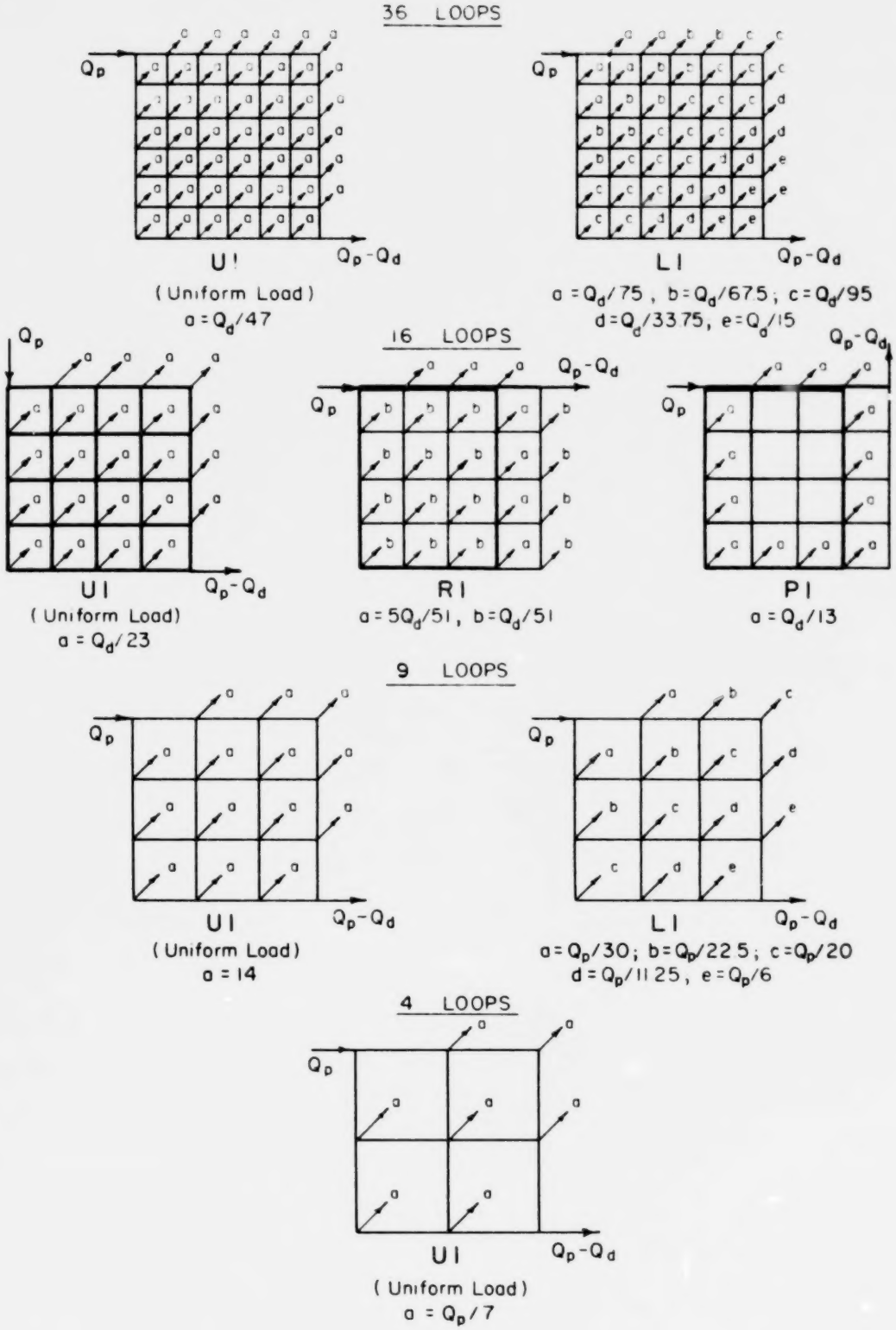
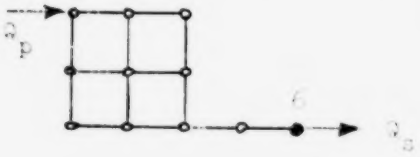


FIG. 11 DEMAND LOAD PATTERNS

and the lower resistance zones respectively closer to the two far corners on the diagonal of network symmetry, and the resistance pattern P which was intended to simulate a particular actual situation. There were four basic demand load patterns (see Figure 11): a uniform load condition U1 in which all of the loads were of equal size; a load pattern L1 in which the network was divided into five zones, with the summation of the demand load for a zone increasing linearly with the relative distance of the zone from the input node; and load patterns R1 and P1 which were used in conjuction with resistance pattern P1, the approximate simulation of an extreme case. As many as six different positions of the storage reservoir were investigated. The junctions used as locations for the storage reservoir are identified in Figure 10 by a number next to the storage node. The storage positions are respectively at the far corner from the input position, the junctions along the diagonal of symmetry between the far corner and the middle of the network, and at one of the corners nearest the input. The sixth position, as shown schematically in Figure 12, is located at the same position as the first point except that there is an extra added resistance of 2k between it and the network.

FIGURE 12

POSITION OF THE SIXTH STORAGE POSITION



EFFECT OF STORAGE LOCATION

Although four different characteristics were varied in the investigation, three of them (resistance pattern, demand load pattern and position of the storage reservoir) have essentially the same effect: they change hydraulically the relative proximity between the input, the hydraulic load center and the storage reservoir. In other words, the relative resistances between these three different elements are changed. An explanation of the effect of changing the relative position of these three elements is simplified if discussion is concentrated on three realms for the position of the storage reservoir relative to the input point and a consolidated demand load output point. For these three realms, the relative position of the storage outlet point is respectively somewhere between the input and the network, somewhere within the distribution network, and beyond the network, Figure 13. The effects from a variation of the storage position within the first and third of these three realms can be determined simply and independently from a consideration of the network. However, the effects of varying the storage position within the distribution network must be explained using complex networks with a spatially distributed load, although some inferences may be made based on the characteristics of the two simpler realms.

The first situation, where the storage is between the pumps and the distribution network, has a rather trivial solution:

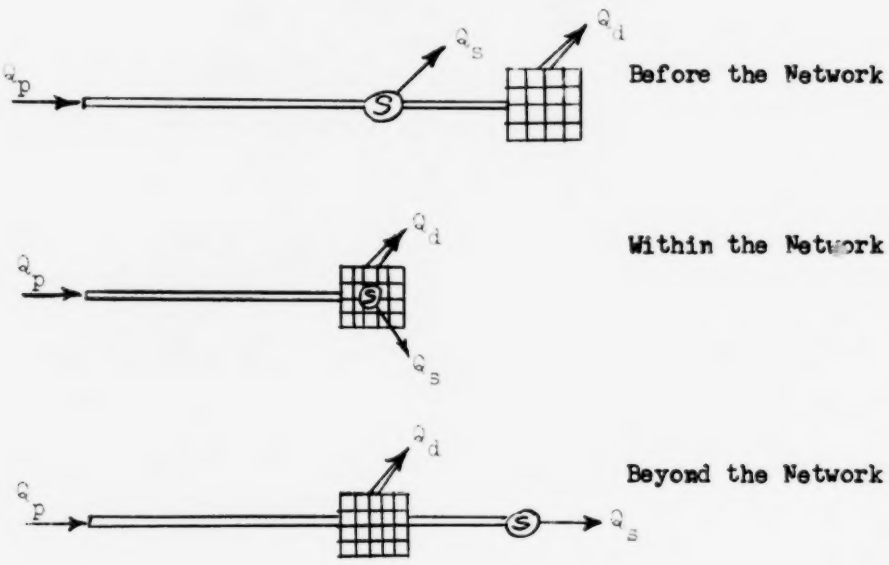
$$\Sigma h/K_o \; Q_d^2 = \left(\frac{Q_p}{Q_d}\right)^2 \qquad (17).$$

Because the entire input at any value of the parameter Q_p/Q_d passes through the line connecting the input point and the storage reservoir outlet point, the solution is synonymous with the conductivity equation for that line, as can be shown by rearranging Eq. (17); moreover, the main part of the distribution system

does not come into consideration. However, this solution is significant in that it is also a limiting case of the second situation with the storage reservoir located within the distribution network.

FIGURE 13

THREE RELATIVE POSITIONS OF THE STORAGE RESERVOIR



The general solution for the simple network in Figure 2 (given by Eq. (10)) is used to demonstrate the effect of varying the hydraulic position of the storage reservoir in the realm beyond the distribution network. For clarification, the network is shown schematically in Figure 14. The total resistance K of the branches between the pump input point and the storage outlet point is held constant, making the resistance pK vary directly with p, which can have any value between 0 and 1.

FIGURE 14

SCHEMATIC REPRESENTATION OF THE NETWORK IN FIGURE 2.

The above simple network has two limiting cases: $p = 1$ and $p = 0$. When $p = 1$, the demand load outlet and the storage outlet points are located at the same position hydraulically. This situation gives the same result as for the previous example. When p is zero, the demand load outlet and the input points are located at the same position hydraulically. Because the flow between the input and the demand outlet has no effect upon the head loss term Σh other than indirectly through the equation $Q_p - Q_d = Q_s$, the location of the storage reservoir has maximum effect upon the solution. The solutions for both limiting cases of $p = 1$ and $p = 0$ are plotted on cartesian coordinate paper and logarithmic coordinate paper in Figures 3 and 4, respectively.

In general, as the storage reservoir is moved further away from the demand load outlet point*, the value of the resistance parameter ϕ decreases, which is what would be expected because the bulk of the flow does not have to move as far hydraulically. The effect upon the exponent N, as determined for Q_p/Q_d equal to 0.75 and 5, is just the opposite: as the storage moves further away, the value of N increases. Because Eq. (16) is not an exact solution, an explanation for the increase in the value of N cannot be clearcut on this particular point. However, in general, the reason for the increase in N is that

*This can also be visualized as the situation of moving the demand load center from the storage outlet towards the input point.

as the demand load comes closer to the input point relative to the storage outlet point, the flow to the demand load has a decreasing effect upon the value of Σh, and the value of Q_p/Q_d at which Σh becomes zero steadily increases. An approximation given by Eq. (16) can be exact only when the value of Σh corresponding to $Q_p/Q_d = 0$ is also zero. For this reason, an increasingly higher value of the exponent N coupled with a lower value of γ is needed to approximate the increasing effect of the constant, γ, and the linear term $\beta(Q_p/Q_d)$ in Eq. (11).

A more convincing demonstration of the effect of varying the relative position of the storage outlet point in the realm beyond the demand load outlet can be given by adjusting the value of γ so that it remains constant while p varies. Using the simple network of Figure 2, this type of variation can be accomplished indirectly by holding the resistance pK constant while increasing the value of K. Although this procedure requires that the solution no longer be dimensionless, it is used since it demonstrates the nature of the increase in the value of N quite well. For this particular situation there is a rotation of the entire plot of the solution on logarithmic coordinate paper about a point where Q_p/Q_d is equal to one (see Figure 15). The reason that there is no change in the value of the head loss parameter Σh at a value of $Q_p/Q_d = 1$ is because at that value there is no flow either to or from storage.

There are two aspects of the multi-loop complex network that tend to make the effects of varying the characteristics significantly different from the effects that would be experienced with the simple networks described. These two aspects are the spatial distribution of the demand load over the entire network instead of at only one outlet, and the grid pattern of the interconnecting branches of the network rather than a one-dimensional branch configuration. In the complex network, the concept of a demand load outlet is replaced by the

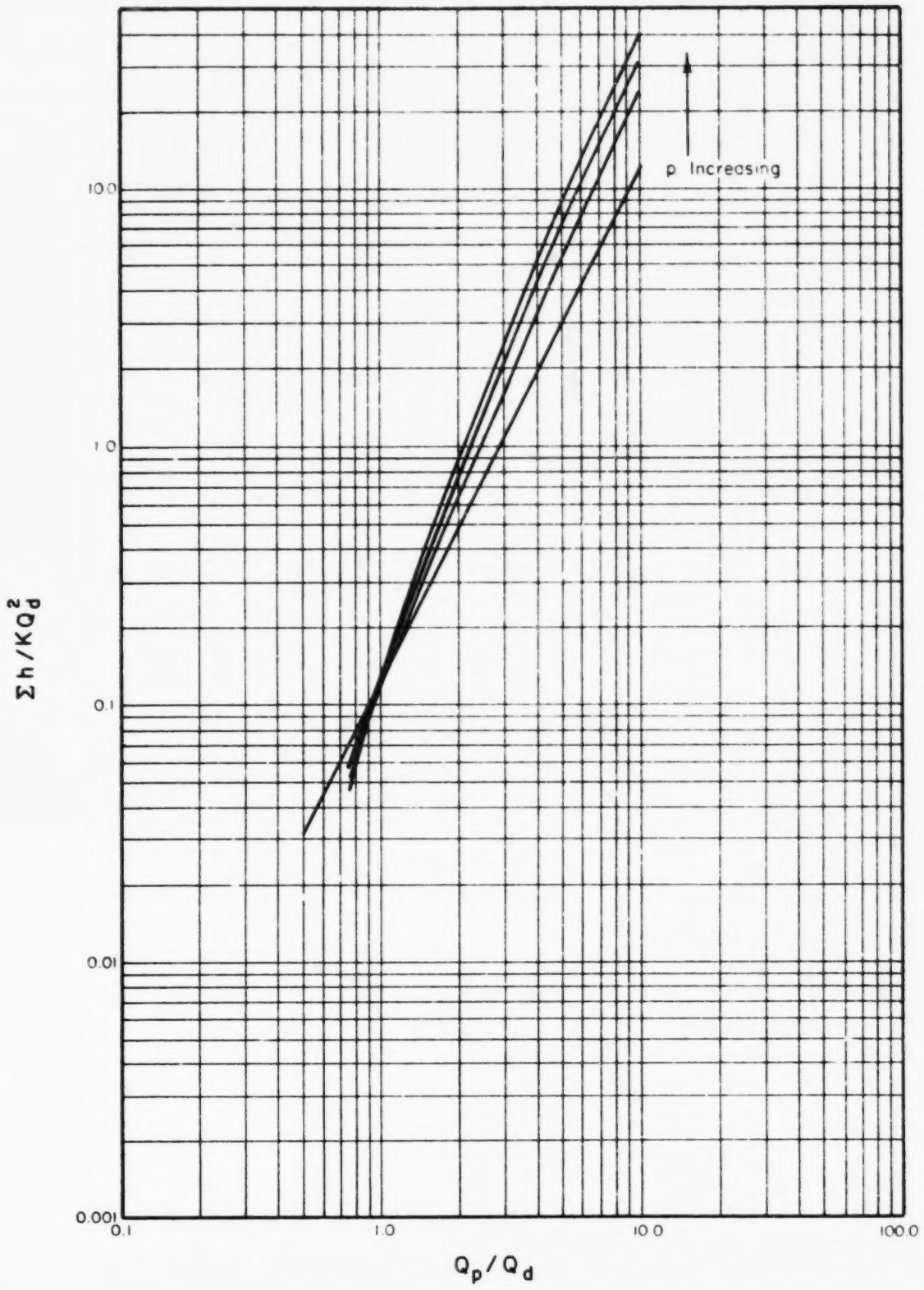


FIG. 15 NETWORK OF FIGURE 2, ϕ CONSTANT, pK CONSTANT, p VARIED.

implicitly defined and somewhat ambiguous concept of a hydraulic load center, which is defined as the approximate point in the network at which the total demand load could be placed without affecting the boundary conditions at the input and storage outlet points. In addition, the concepts of hydraulic position and relative resistance become more difficult to reconcile due to their implicit nature in the complex network. Thus, it is now necessary to discuss variations within limits and in terms of generalities.

The effect of varying the position of the storage reservoir relative to the input point and the hydraulic load center is best demonstrated by physically moving the storage output point across the network. The results for the simple networks give a good indication of what would be expected in the complex network for the two extreme limits of the relative position of the storage reservoir outlet: (1), with the storage located at the input side of the spatially distributed load, and (2), with the storage located at a relatively great hydraulic distance, approaching infinity, from the input and demand load outlet points. A 36-loop network with a uniform resistance pattern and a uniform demand load pattern (36-U1-U1-S series) is used to demonstrate the effect of moving the storage reservoir position across a complex network. Five different positions of the storage reservoir are used, starting with a point completely outside the network (position 6 connected to position 1, at the far corner of the network, by a branch with a resistance value of 2k) and then progressively inwards along the diagonal of symmetry towards the input point.

Table 2 gives the resulting variations of Φ and N. As may be noted, the results are similar to those for the simple network of Figure 2 in that the value of the exponent N decreases towards a value of 2.0 (and the value of Φ increases) as the storage position is moved across the network towards the input point. Similar results for the 16- and 25-loop networks in Table 2 reinforce this conclusion.

Except for the case 36-U1-U1-S6, the largest N were obtained with the P1 resistance pattern, with storage at S5, Table 2. This resistance pattern was used to determine how large N might be. However, it was necessary to use position S6 to raise N above 3.

TABLE 2

CHANGES IN Φ AND N WITH THE STORAGE POSITION VARIED

Network	Storage Position	N	Φ
36-U1-U1-	S6	3.565	0.01108
36-U1-U1-	S1	2.369	0.02975
36-U1-U1-	S2	2.175	0.03653
36-U1-U1-	S3	2.132	0.04528
36-U1-U1-	S4	2.111	0.05942
16-U1-U1-	S1	2.379	0.04108
16-U1-U1-	S2	2.170	0.05592
25-U1-U1-	S1	2.373	0.03447
25-U1-U1-	S2	2.174	0.04405
25-U1-U1-	S3	2.128	0.05798
16-P1-R1-	S5	2.732	0.11940
16-P1-P1-	S5	2.628	0.14260
16-P1-U1-	S5	2.846	0.09820

EFFECTS OF LOADING AND RESISTANCE PATTERNS

Variations of the other two network characteristics, demand load pattern and resistance pattern, had considerably less effect on the value of the exponent N, although the nature of the effect was somewhat surprising in the case of the resistance patterns. The only load pattern variation used other than the uniform load pattern was pattern L1, with the bulk of the load closer to the storage reservoir, located at the far corner of the network. From the results for the simple network, one would predict that the value of N would decrease slightly and the value of Φ would increase. This prediction is substantially verified, although the decrease in the value of the exponent N is insignificant, especially for the larger network, Table 3.

TABLE 3

CHANGES IN Φ AND N WITH
DEMAND LOAD PATTERN VARIATION

Network	N	Φ
36-U1-U1-S1	2.369	0.02975
36-U1-L1-S1	2.364	0.03170
9-U1-U1-S1	2.384	0.05109
9-U1-L1-S1	2.343	0.05560

With the resistance K between the input point and the storage outlet point as defined for the complex networks, summing the resistances over a particular route between the input and the storage reservoir results in an arbitrary evaluation of K, especially for a network that has many different relative values for the resistances of its branches. This characterization also makes the prediction of relative variations in the value of the exponent N

rather difficult. This point is brought out rather clearly in the results obtained by varying the resistance patterns.

Using the thirty-six loop network several resistance patterns were investigated (L1, M5, M25, N5, N25, and O5), while keeping the demand load pattern uniform. With the exception of the L1 resistance pattern, all resistance patterns are symmetrical with respect to the geometric center of the network; and none change the relative hydraulic position of the three primary elements of the network due to a shift in the relative resistances between these elements.

The results for the different resistance patterns are given in Table 4. The value of the exponent N increases slightly with an increase in the ratio of the largest to the smallest branch resistance. The corresponding increase in ϕ would be entirely expected.

TABLE 4

EFFECT ON N AND ϕ WITH VARIATION IN RESISTANCE PATTERN

36 Loop Network With Uniform Demand Load Pattern (U1) and Tank at S1

Resistance Pattern	N	ϕ	Ratio of Largest to Smallest Branch Resistance
U1	2.369	0.02975	1
L1	2.433	0.03962	4.97
N5	2.387	0.03630	5
N25	2.398	0.04186	25
M5	2.397	0.03836	5
M25	2.418	0.04749	25
O5	2.411	0.03944	5

16 Loop Network With Uniform Demand Load Pattern (U1) and Tank at S1

Resistance Pattern	N	ϕ	Ratio of Largest to Smallest Branch Resistance
U1	2.379	0.04108	1
M5	2.403	0.04853	5
M25	2.418	0.05503	25
O5	2.425	0.05089	5
O25	2.468	0.06267	25

EFFECT OF SIZE OF NETWORK

The primary characteristic remaining to be discussed is the variation in the overall size of the network. For this particular variation the size of the network is increased while all other characteristics are held constant. This variation was investigated for the uniform networks (series-U1-U1-S1), several of the large-small branch resistance grid pattern networks, and a network with both a nonuniform load pattern (L1) and a nonuniform resistance pattern (L1). Although there was a slight increase in the value of the exponent N with a decrease in the number of loops for all cases, the increase was very slight (see Table 5 and Figure 16). On the other hand, the value of the constant Φ was increased significantly with a decrease in the number of loops as would be expected, because there are fewer branches to accommodate the flow in the smaller networks.

TABLE 5

EFFECT OF VARYING NETWORK SIZE

Network	N	Φ
36-U1-U1-S1	2.37	0.02975
25-U1-U1-S1	2.37	0.03447
16-U1-U1-S1	2.38	0.04108
9-U1-U1-S1	2.38	0.05109
4-U1-U1-S1	2.38	0.06900
36-M25-U1-S1	2.42	0.04749
16-M25-U1-S1	2.42	0.05503
36-L1-L1-S1	2.34	0.04574
9-L1-L1-S1	2.37	0.09369

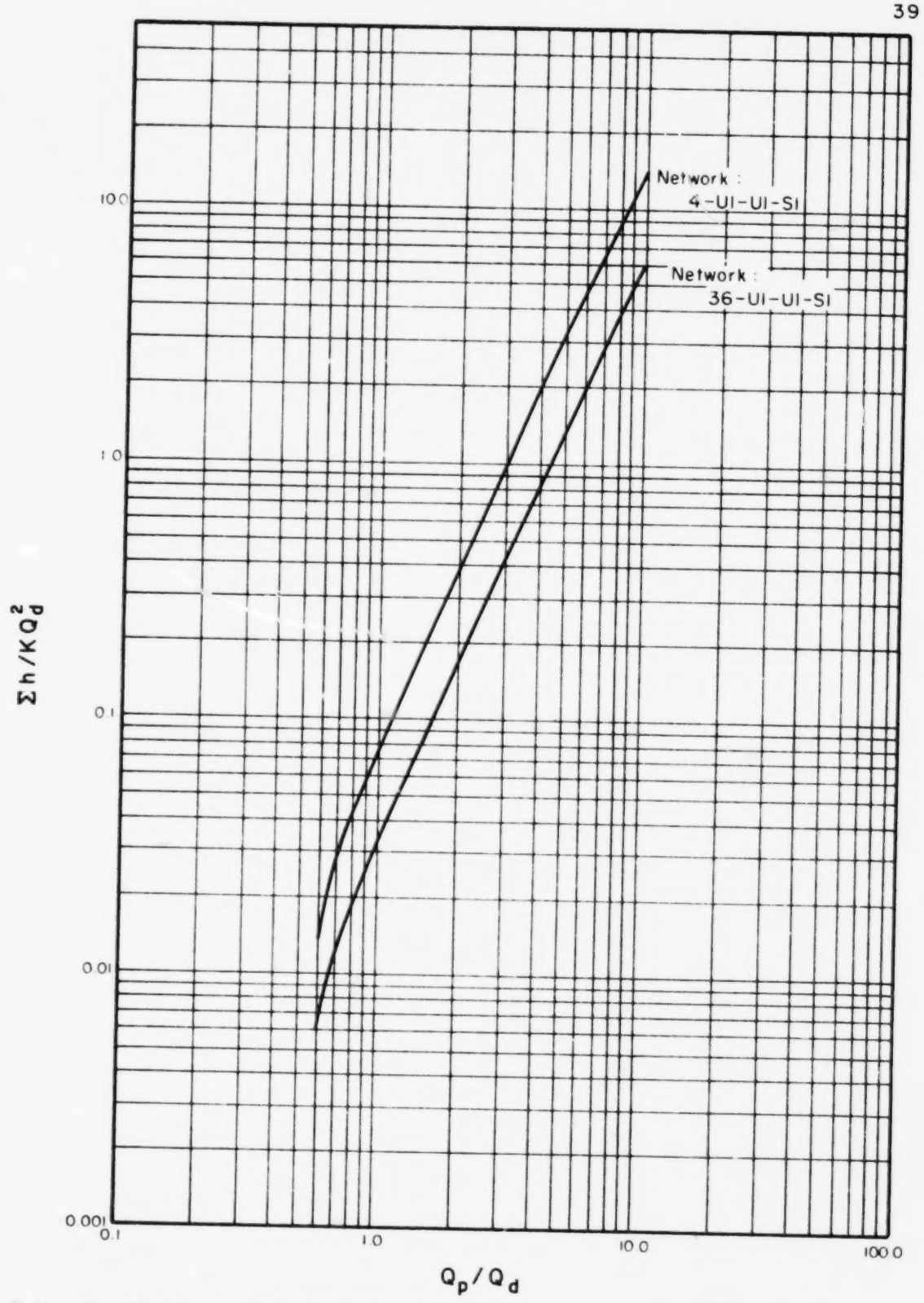


FIG 16 EFFECT OF VARIATION IN NETWORK SIZE

CONCLUSIONS

The variation between $\Sigma h/Q_d^2$ and Q_p/Q_d has been shown to be only partially satisfied by Eq. (2), using a series of idealized water distribution networks under various combinations of branch resistance, load pattern, network size and equalizing storage location. Provided that at least three network balances are performed for actual network studies to insure a realistic evaluation of Σ and N, Eq. (2) can be used as an expedient.

A closer replication could be assured by using a polynomial interpolation subroutine in computer system balances, for interpolating between the results for only about a half-dozen network balances. While this refinement would be justified in a final design, use of Eq. (2) for studies of alternative system components in preliminary design should be adequate if judiciously applied. The unsuitability of Eq. (2) for some of the idealized networks studied may be an exaggeration inasmuch as it has been found adequate for some actual systems as the result of testing against field measurements. For an operation simulation study, such as with a year of hourly demands, Eq. (2) should be adequate if properly applied. Subjective judgment is required in the employment of any approximation.

In a companion study of equalizing storage requirements[3], the value of N for m = 2 was varied between 2 and 4. From the results of the present study it is evident that an N as high as 4 requires that the elevated storage be located opposite and quite remote from the center of demand load on a separate feeder main. This would be a rather unique circumstance.

The variation of Σ with various changes in network conditions followed expectations. It was not anticipated that N would vary only between about 2 and 2.5 for rather drastic changes in network conditions, and this range of N might well be typical of a number of actual arterial water distribution system networks.

The network balances performed for this study were made using the IBM 7094 digital computer of the University of Illinois. The first author wrote the program used and conducted the study reported. Of the 32 series of runs made, the results of 28 have appeared in the preceding text, in terms of Φ and N. Results for individual runs are given in Appendix III.

REFERENCES

1. McPHERSON, M. B. - "Generalized Distribution Network Head-Loss Characteristics". Trans. ASCE, 126:Part I:1190 (1961).

2. McPHERSON, M. B. - "Applications of System Analyzers: A Summary". Proc. Fourth San. Conf., Water Distribution Systems, Univ. of Illinois Engrg. Exper. Sta. Circular No. 75:23 (August, 1962). Reprinted in Water and Sewage Works, 109:Reference Number:R-53 (November 30, 1962).

3. McPHERSON, M. B., and PRASAD, R. - "A Study of Distribution System Equalizing Storage Hydraulics". Civil Engrg. Studies, Hydraulic Engrg. Series No. 6, University of Illinois, Urbana, March, 1965.

4. McPHERSON, M. B., and RADZIUL, J. V. - "Water Distribution Design and the McIlroy Network Analyzer". Proc. ASCE, 84:No. HY2:Paper 1588 (April, 1958).

5. McPHERSON, M. B., and PRASAD, R. - "A Study of Power Consumption for Equalizing Storage Operating Options". Civil Engrg. Studies, Hydraulic Engrg. Series No. 8, University of Illinois, Urbana, June, 1965.

6. CROSS, HARDY - "Analysis of Flow in Networks of Conduits or Conductors". Engrg. Experiment Station Bull. No. 286, Vol. XXXIV, No. 22, University of Illinois, November 13, 1936.

7. FAIR, GORDON M., and GEYER, JOHN C. - Water Supply and Waste Water Disposal. John Wiley and Sons, Inc., N.Y., N.Y., 1954.

A DEMONSTRATION OF SOLUTION UNIQUENESS IN NETWORK ANALYSIS

R. A. Wiseman

The procedure used in carrying out this proof might appear inadequate to a mathematician; however, if after reading through the following presentation the reader is convinced that one and only one solution exists for a network with a particular set of boundary conditions the presentation has served its purpose.

The basic procedure used is essentially the procedure of Birkhoff and Diaz[A-1]* when they proved the uniqueness of a solution for a general network situation; however, this proof is concerned more with the specific problem of a water distribution network.

The network defined in this proof has somewhat less stringent restrictions imposed on it than the mathematical model defined in the main text of this report. For this reason certain parts will have to be redefined.

The water distribution network is defined, as in the text, as a system of interconnected nodes and branches. The nodes form a set N* consisting of all the nj nodes, and the branches form a set L of nb branches. The connectivity of these nodes and branches is defined by the matrix $|A|$, where the element a_{ij} indicates the incidence of the jth branch upon the ith node. (Refer to the main text for a more complete definition of the nature of a_{ij} and $|A|$.) As previously mentioned in the text, there are only two necessary conditions that need be

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*D'Auriac[A-2] and Duffin[A-3] were also concerned with the uniqueness of a network flow solution. In addition, both Birkhoff and Diaz considered the problem of proving the existence of a solution. Birkhoff and Diaz, in addition to proving the existence of an exact solution, proved the existence of a solution when iterative convergence techniques are used. Oldenburger[A-4] was concerned with convergence of linear resistance networks, and Warga[A-5] indicated the existence of a solution for non-linear resistance networks when iterative techniques are used.

imposed upon the connectivity of the network: (1), the network must be completely interconnected so that there are no subnetworks not connected directly to other parts of the network; (2), no more than one branch may connect any pair of two nodes directly. Neither of these two conditions are especially difficult to satisfy. If a particular system does not meet the first condition because it has several separate, unconnected networks within itself, this condition may be satisfied by considering the parts separately. If the second restriction is violated, the network can be modified by introducing an equivalent single branch resistance to replace the two or more branches that originally connected the pair of nodes.

Because a certain number of the nodes that comprise the set N* communicate with conditions outside the network, they are defined as boundary nodes and are assigned to the subset E. The function of these nodes on the boundary is to connect the system to external boundary conditions which provide an influx to the system in conjunction with a certain head at some boundary node or nodes and receive an efflux at others. The internal nodes, which are the remaining nodes in the set N*, define a second subset I.

In the idealized model used in the main part of the report, the conductivity function for flow in a pipe was assumed to be exponential in nature (Eq. (3)). As far as proving the uniqueness of a flow solution is concerned, this restriction is not necessary. There are only two conditions that a conductivity function must satisfy as far as this proof is concerned: (1), it must be continuous; (2), it must be nondecreasing, i.e., there can

be no peak flow. These conditions are given by

$$Q_j = q_j\ (h_j) \qquad \text{(A-1)},$$

and illustrated in Figure A-1.

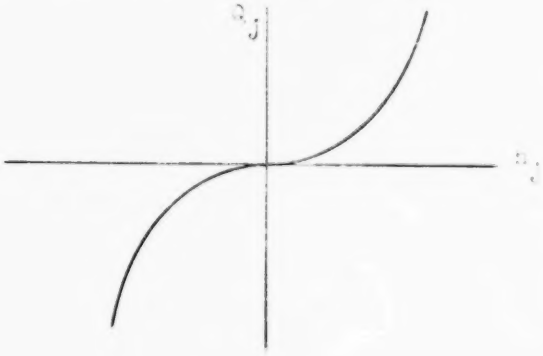

FIGURE A-1

THE CONDUCTIVITY RELATIONSHIP
$Q_j = q_j\ (h_j)$

The requirements for conservation of flow and a single-valued pressure at each and every node are still regarded as the governing conditions imposed upon the flow solution for a network regardless of the nature of its function. A short discussion of the flow conservation condition is necessary to clarify certain aspects peculiar to a network. Consider the conservation condition as applied to the set N*:

$$D_i = \sum_{j=1}^{n} a_{ij} Q_j \quad \text{for } i = 1,2,\ldots nj \qquad \text{(A-2)}.$$

By definition, the influx D_i does not exist on any of the "interior" nodes comprising the set I; and it can be concluded that

$$\sum_{j=1}^{nb} a_{ij}Q_j = D_i = 0 \quad \text{for all i in the set I}$$

and* $$\sum_{I} D_i = 0 \qquad \text{(A-3)}.$$

In essence, the water distribution network problem can be considered as being a "boundary value problem" because certain specified conditions are imposed upon the system at the boundary nodes. Both D'Auriac[A-2] and Birkhoff and Diaz[A-1] considered three possible boundary value problems:

I. The head is given at all nodes on the boundary (Figure A-2a);

II. The influx is given on the boundary (Figure A-2b);

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*Another aspect, not particularly relevant to the proof, is an outgrowth of this conclusion. Due to the nature of the matrix [A] (Each branch is incident upon two nodes: one in a positive sense and the other in a negative sense),

$$\sum_{I}\sum_{J} a_{ij}Q_j = 0 \ .$$

Substitution of Eq. (A-2) into this relationship gives

$$\sum_{I} D_i = 0$$

and consequently

$$\sum_{I} D_i = 0$$

which in effect shows that if there are no leaks within the network the input must equal the output.

III. A functional relationship between head and influx is given for the boundary nodes. Of necessity, one restriction must be imposed upon this functional relationship: it must be nonincreasing in nature. As far as either pumps or normal outlet facilities are concerned, this condition is completely compatible with their flow-head relationships (Figure A-2c and Figure A-2d).

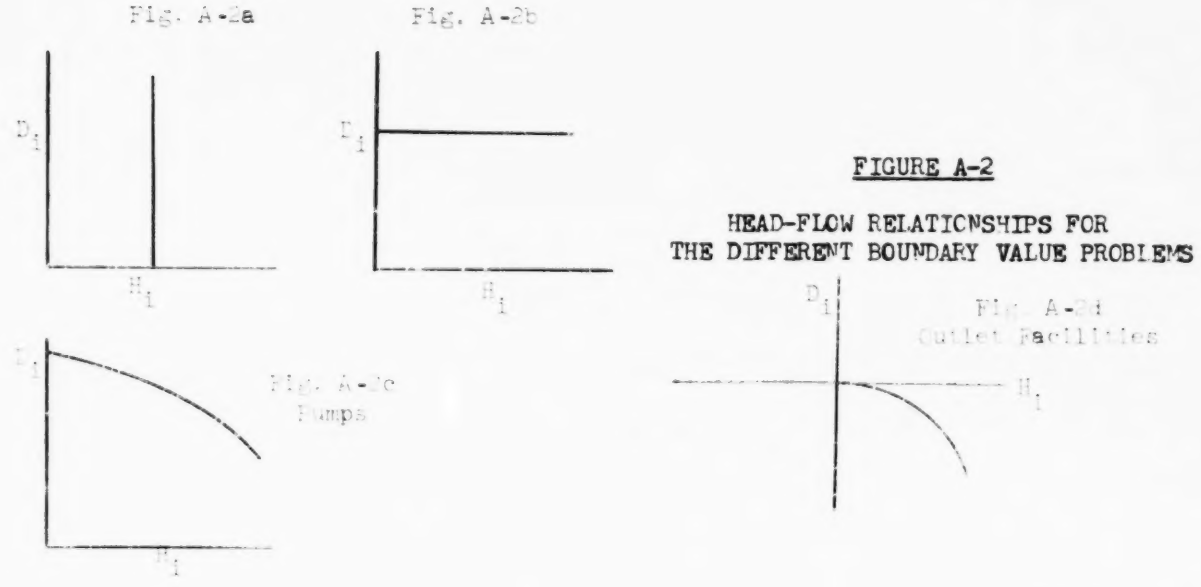

FIGURE A-2

HEAD-FLOW RELATIONSHIPS FOR THE DIFFERENT BOUNDARY VALUE PROBLEMS

The procedure used to prove that there is only one possible flow solution for a network with a particular set of boundary conditions is to prove that a hypothetical second solution cannot possibly exist. In brief this proof consists of: (1), assuming a second flow solution (say, H_1' and $q_j(h_j')$) in addition to the known solution (H_1 and $q_j(h_j)$); (2), defining a fictitious energy term X based on the differences between the two solutions in terms of the internal branch flows ($Q_j - Q_j'$) and the branch head differentials

$(h_j - h'_j)$; (3), deriving an expression for X in terms of the heads and influxes at boundary nodes; (4), proving that the two equations defining X internally and on the boundary are consistent if and only if $X \equiv 0$; and (6), proving that if $X \equiv 0$ the head term $(H_1 - H'_1)$ must be zero for all i in the set N*.

On the assumption that there is a second solution (H'_j for j = 1,2..nb) consistent with the given boundary conditions in addition to the known solution (H_j for j = 1,2,..nb), an energy function X is defined internally over the set L of nb branches:

$$X = \sum_{j=1}^{nb} \left\{ \left[q_j(h_j) - q_j(h'_j) \right] (h_j - h'_j) \right\} \qquad \text{(A-3)}$$

where

$$(h_j - h'_j) = \sum_{i=1}^{nj} a_{ij}(H_i - H'_i) \quad \text{for } j = 1,2,\ldots nb.$$

Due to the nondecreasing nature of the conductivity relationship (Eq. (A-1)), each term in the series on the right side of Eq. (A-3) must always be either positive or zero. Correspondingly, $X \geq 0$. If X is equal to zero, the difference $(H_1 - H'_1)$ must be equal to a constant value for all the nodes in the set N*.

An expression for X can be derived from Eq. (A-3) such that X is defined on the boundary. Rearranging Eq. (A-3),

$$X = \sum_{i=1}^{nj} \sum_{j=1}^{nb} \left\{ \left[a_{ij} q_j(h_j) - a_{ij} q_j(h'_j) \right] \left[H_i - H'_i \right] \right\}$$

$$= \sum_{i=1}^{nj} \left\{ \left[\sum_{j=1}^{nb} (a_{ij} q_j(h_j)) - \sum_{j=1}^{nb} (a_{ij} q_j(h'_j)) \right] \left[H_i - H'_i \right] \right\} .$$

Substituting Eq. (A-2) into the above equation gives

$$X = \sum_{i=1}^{n} \left\{ \left[D_i - D_i' \right] \left[H_i - H_i' \right] \right\} \qquad \text{(A-4)}.$$

However, because $\sum_{I} D_i = 0$ and $\sum_{I} D_i' = 0$, they can be subtracted from the above equation without changing the value of X. The removal of the nodes of the set I leaves only the boundary set E to be considered:

$$X = \sum_{E} \left\{ \left[D_i - D_i' \right] \left[H_i - H_i' \right] \right\} \qquad \text{(A-5)}.$$

The three different boundary value problems are considered separately:

I. For the first boundary value problem, $X \equiv 0$, because H_i is given as a constant on the boundary ($H_i - H_i' = 0$ for all i in the subset E).

II. For the second boundary value problem, $X \equiv 0$, since the influx (or efflux) is defined as a constant on the boundary ($(D_i - D_i') = 0$).

III. In the third boundary value problem, D_i was required to be a nonincreasing function of the node head H_i. For this reason, each individual term in the series on the right side of Eq. (A-5) must be either zero or negative; therefore, $X \leq 0$. Comparing this result with the internal definition of X which restricts X to values that are either zero or positive, it is again deduced that $X \equiv 0$. Also, because the individual terms of the series vanish identically, the difference $(H_i - H_i')$ must become zero for all nodes on the boundary.

Consideration of the first and third boundary value problems indicates that $H_i - H_i' = 0$ for all nodes on the boundary. Comparison of this condition with the condition that for $X = 0$, $H_i - H_i' =$ constant for all nodes in the set N^*, gives the result that $H_i - H_i' = 0$ for $i = 1,2...nj$ for these two particular boundary value problems. Thus the hypothetical solution H_i' for $i = 1,2,...nj$ is identical with the known solution for these two sets of conditions. In the second boundary value problem, $H_i - H_i'$ is equal to a constant for $j = 1,2,...nj$. Because the conductivity function is nondecreasing in nature, the flow solution Q_j' must be identical with the known solution Q_j for all branches in the set L, even though the node heads may differ by a constant value over the entire network.

REFERENCES FOR APPENDIX I

A-1 Birkhoff, Garrett and Diaz, J. B., "Nonlinear Network Problems", Quarterly of Applied Mathematics, pp. 431-443, v. 13, n. 4, 1956.

A-2 D'Auriac, Angles, "A Propos De Solution Dans Les Problemes De Reseaux", La Houille Blanche, pp. 209-211, May-June, 1947.

A-3 Duffin, R. J., "Nonlinear Networks. IIa", Bulletin of the American Mathematics Society, pp. 963-971, v. 53, 1947.

A-4 Oldenburger, R., "Convergence of Hardy Cross's Balancing Process". Jour. of Applied Mechanics, Dec., 1940.

A-5 Varga, Jack, "Determination of Steady-State Flows and Currents in a Network", Instrument Society of America Proceedings, Vol. 9, December, 1954.

APPENDIX II

A STUDY OF ALTERNATIVE EXPRESSIONS DESCRIBING GENERALIZED DISTRIBUTION NETWORK HEAD LOSS CHARACTERISTICS

by

MacKenzie L. Davis,
with a
private communication from
M. H. Diskin

Pipe Network Solution

M. H. Diskin*

1. Series Pipe Network

Consider a series pipe network defined as a network of pipe segments AB, BC, CD, ... NM, placed in series (Fig. 1), where each segment may be composed of a number of pipes of varying diameters and roughness placed either in series or in parallel or both. Let it be stipulated that water can be withdrawn from the network only at points A, B, C, D, ... M separating the segments.

If the network is proportionally loaded then the outflow at each point may be represented as a constant fraction of the total demand Q_d. Let the fraction at each point be aQ_d, bQ_d, etc., where $0 < a < 1$, $0 < b < 1$, etc. and $\Sigma[a + b + c + \ldots + n + m] = 1.00$.

Let the inflow pumped to the network at point A be Q_p and the outflow to storage at point M be Q_s; then by continuity

$$Q_p = Q_d + Q_s$$

If the pipe segments are numbered 1, 2, 3, etc. then the flow in each pipe is given by

$$Q_1 = Q_p - aQ_d = Q_p - \alpha Q_d$$
$$Q_2 = Q_p - (a + b)Q_d = Q_p - \beta Q_d$$
$$Q_3 = Q_p - (a + b + c)Q_d = Q_p - \gamma Q_d$$

where $\alpha = a$, $\beta = (a + b)$, $\gamma = (a + b + c)$ etc. and $0 < \alpha < \beta < \gamma < \ldots < 1.0$

(*) Private Communication to M. B. McPherson, June, 1963).

The head losses in each pipe segment are given by

$$h_1 = K_1 Q_1^m = K_1 (Q_p - \alpha Q_d)^m$$
$$h_2 = K_2 Q_2^m = K_2 (Q_p - \beta Q_d)^m$$
$$h_3 = K_3 Q_3^m = K_3 (Q_p - \gamma Q_d)^m$$
etc.

The head loss between any two points in the system is given by

$$\Sigma h = \Sigma h_i$$

with the summation carried over all pipe segments between the two points in question

$$\Sigma h = \Sigma K_i (Q_p - \lambda_i Q_d)^m$$

where K_i is the sum of the values of the resistance coefficients for all segments involved When Σh is taken across the entire network, $\lambda_i = 1$

Taking Q_d^m out of the summation

$$\Sigma h = Q_d^m \Sigma K_i \left(\frac{Q_p}{Q_d} - \lambda_i\right)^m$$

For a given network and loading K_i and λ_i are constant so the last equation shows that

$$\left(\frac{\Sigma h}{Q_d^m}\right) = \Sigma K_i \left(\frac{Q_p}{Q_d} - \lambda_i\right)^m = \psi\left(\frac{Q_p}{Q_d}\right)$$

which means that the losses between any two points divided by the demand flow to the power m, (Q_d^m), is a function of the ratio $(\frac{Q_p}{Q_d})$

As a first approximation of the expansion

$$\left(\frac{Q_p}{Q_d} - \lambda_i\right)^m = \left(\frac{Q_p}{Q_d}\right)^m - m\left(\frac{Q_p}{Q_d}\right)^{m-1} \lambda_i +$$

so that

$$\left(\frac{\Sigma h}{Q_d^m}\right) = \Sigma K_i\left(\frac{Q_p}{Q_d} - \lambda_i\right)^m = \left(\frac{Q_p}{Q_d}\right)^m \Sigma\left(K_i\right) - m\left(\frac{Q_p}{Q_d}\right)^{m-1} \Sigma\left(K_i\,\lambda_i\right)$$

or

$$\frac{\Sigma h}{Q_d^m} = c_1\left(\frac{Q_p}{Q_d}\right)^m - c_2\left(\frac{Q_p}{Q_d}\right)^{m-1}$$

Even if the expression $\left(\frac{Q_p}{Q_d} - \lambda_i\right)^m$ is expanded to more terms the form of the equation obtained for $\left(\frac{\Sigma h}{Q_d^m}\right)$ will be a series of terms of power of $\left(\frac{Q_p}{Q_d}\right)$ establishing the fact that:

$$\frac{\Sigma h}{Q_d^m} = \Psi\left(\frac{Q_p}{Q_d}\right)$$

Considering again the equation

$$\frac{\Sigma h}{Q_d^m} = c_1\left(\frac{Q_p}{Q_d}\right)^m - c_2\left(\frac{Q_p}{Q_d}\right)^{m-1}$$

it can be written as

$$\frac{\Sigma h}{Q_d^m} = \left(\frac{Q_p}{Q_d}\right)^m \left[c_1 - \frac{c_2}{\left(\frac{Q_p}{Q_d}\right)}\right]$$

or

$$\left(\frac{\Sigma h}{Q_p^m}\right) = c_1 - c_2\left(\frac{Q_d}{Q_p}\right)$$

so that for a first approximation there is a <u>linear relationship</u> between $\left(\frac{\Sigma h}{Q_p^m}\right)$ and $\left(\frac{Q_d}{Q_p}\right)$. (Note the change in variables from the equation proposed by M B McPherson, A.S.C.E. Transactions, Vol 126, Part I 1961)

TABLE I

COMPARISON OF PIPE NETWORK SOLUTIONS

					Head Loss Σh from Pumping Station to Storage						
								M H Diskin			
No	Q_d	Q_p	Q_p/Q_d	Q_d/Q_p	Meas	M B M	Difference	$Q_d/Q_p > 1$	Difference	$Q_d/Q_p < 1$	Difference
1	52 5	35 01	0 667	1 50	18 0	19 3	1 3	18 16	0 16	c_1 = 04443	
2	52 5	35 70	0 680	1 47	19 1	20 3	1 2	19 2	0 1	c_2 = 00158	
3	63 0	47 25	0 750	1 33	36 1	36 1	0 0	36 46	0 36		
4	52 5	39 38	0 750	1 33	26 7	25 8	0 9	26 03	0 67		
5	42 0	31 50	0 750	1 33	17 6	17 1	0 5	17 22	0 38		
6	42 0	35 53	0 846	1 18	24 2	23 0	1 2	24 22	0 02		
7	52 5	47 51	0 905	1 105	42 3	40 9	1 4	43 76	1 46		
8	47 2	47 2	1 00	1 00	44 2	43 0	0 8	46 38	2 18	53 52	9 32
9	43 5	43 5	1 00	1 00	38 2	36 9	1 3	39 89	1 69	46 02	7 82
10	39 4	39 4	1 00	1 00	31 9	30 8	1 1	33 23	1 33	38 34	6 44
11	35 5	35 5	1 00	1.00	26 7	25 4	1 3	27 40	0 70	31 61	4 91
12	31 5	31 5	1 00	1 00	21 2	20 3	0 9	21 96	0 76	25 37	4 17
13	37 5	47 13	1 25	0 80	53 8	49 5	4 3	c_1 = 06144		53 73	0 07
14	31 5	39 38	1 25	0 80	36 9	35 2	1 7	c_2 = 0243		38 15	1 2
15	25 1	31 38	1 25	0 80	24 8	23 5	0 7			25 35	0 55
16	18 9	23 63	1 25	0 80	14 1	13 7	0 4			15 00	0 9
17	18 9	28 35	1 50	0 667	21 1	21 4	0 3			21 31	0 21
18	28 4	47 43	1 67	0 598	59 5	59 5	0 0			54 84	4 66
19	23 6	39 41	1 67	0 598	42 0	42 1	0 1			38 92	3 08
20	18 9	31 56	1 67	0 598	27 8	27 9	0 1			25 81	1 99

Data from A S C E Transactions Paper No 3229, p 1198
McPherson's ℓ = 0 0344, n = 2 46 m = 1 85 (TORR H S - P S to Storage)

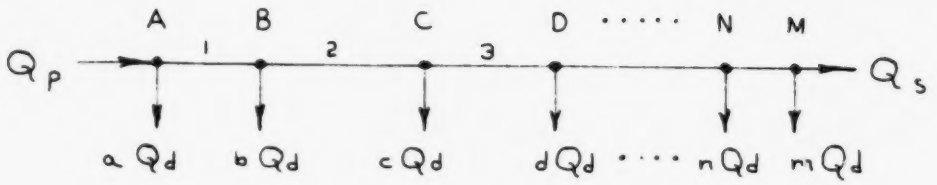

Figure 1

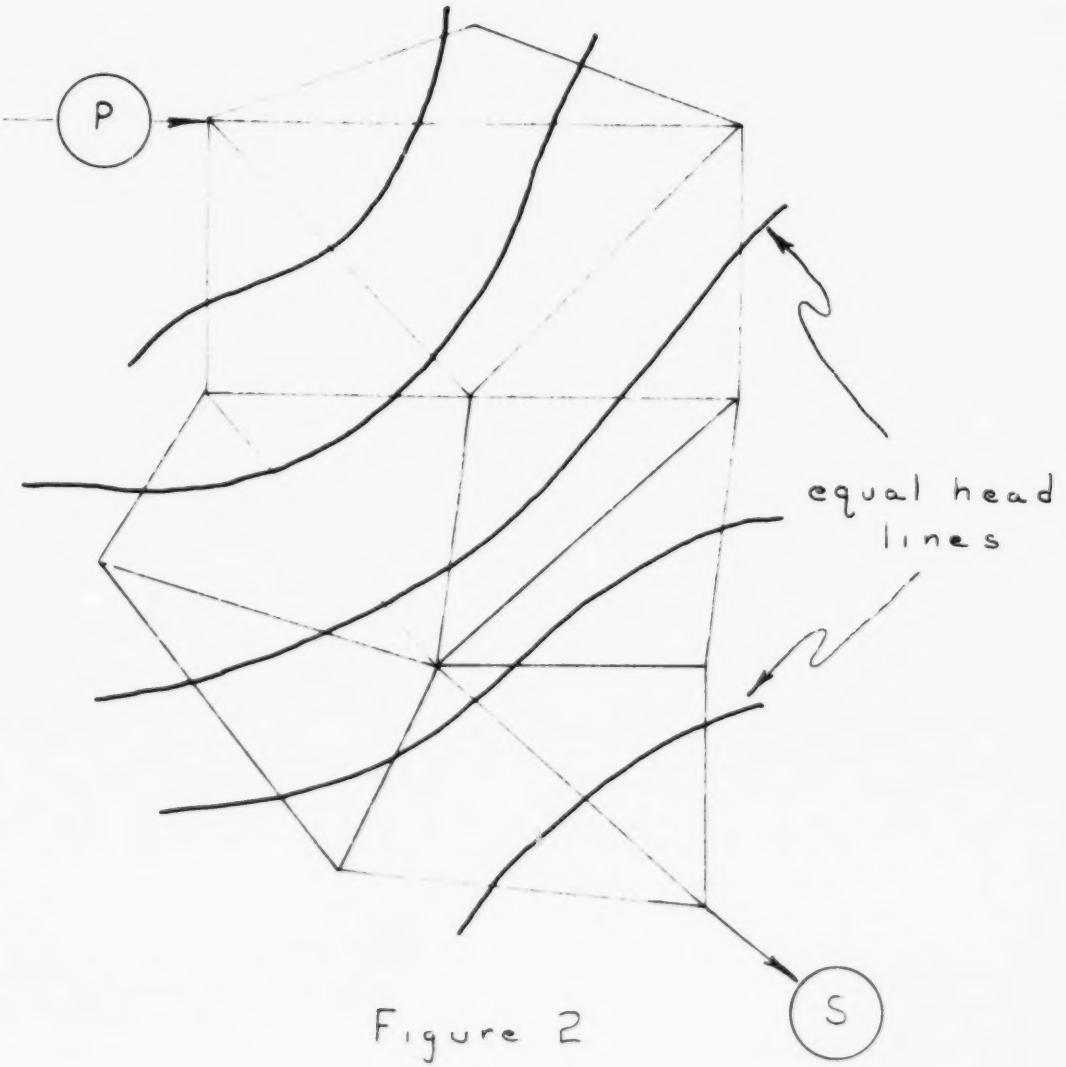

Figure 2

Report on the Analysis of M. H. Diskin's Pipe Network Solution

Mackenzie L. Davis

The analysis corroborated the mathematical soundness of Mr. Diskin's derivation. None of his assumptions imposed conditions which could be deemed inconsistent with commonly used design assumptions.

The numerical analysis of the validity of the pipe network solution was carried out in two phases. In the first phase a comparison of the results of Mr. Diskin's solution, Prof. M. B. McPherson's solution[1], and the actual measurements was made. Tables I, II, III and IV are the tabulations of the comparison. The 'first approximation' of M. Diskin's solution was used as shown below.

$$\Sigma h = \left[c_1 - c_2 \left(\frac{Q_d}{Q_p} \right) \right] Q_p^{\,m}$$

[Two points were chosen from each set of data to solve for each respective c_1 and c_2. When it was necessary a c_1 and a c_2 was computed for each of the conditions $\frac{Q_d}{Q_p} > 1$ and $\frac{Q_d}{Q_p} < 1$.

Because two runs were required to solve for the constants for each range ($\frac{Q_d}{Q_p} > 1$ and $\frac{Q_d}{Q_p} < 1$), a choice of runs in addition to those used by Prof. McPherson in his solution for ℓ and n, was required.

Each range was extended to include $\frac{Q_d}{Q_p} = 1$. The values at this overlap show that extension of boundary conditions to this transition region is not sound.

All the calculations were made by hand with the aid of logarithms.]

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1. A.S.C.E. Transactions Paper No. 3229, Vol. 126, Part I, 1961

In each tabulation Prof. McPherson's solution showed a lesser tendency to deviate from the measured values than Mr. Diskin's solution. Using the results tabulated in Table I the standard deviation of each method was computed. ($\sigma = \sqrt{\frac{(x-m)^2}{n-1}}$ where x = difference, M = mean difference and n = number of points) The standard deviation for Prof. McPherson's solution was 0.938 ft. and for Mr. Diskin's solution (using the best series of results) it was 1.145 ft.

An expansion of the 'first approximation' was used in the second phase of the numerical analysis to assess the possibility of increased accuracy. The expansion is shown below.

$$\Sigma h = \left[c_1 - c_2\left(\frac{Q_d}{Q_p}\right) + c_3\left(\frac{Q_d}{Q_p}\right)^2 - c_4\left(\frac{Q_d}{Q_p}\right)^3 \right] Q_p^m$$

[This form permits consideration only of cases where λ_i = constant. The constants were determined for $\left(\frac{Q_d}{Q_p}\right) > 1$ (Table Ia) for Table I only because of little difference from the original expansion. All the calculations were made by hand with the aid of logarithms.]

The expansion did not favorably alter the answers developed from the 'first approximation'. An explanation for this lack of positive response lies in the method of determining the constants. Because the constants are determined from simultaneous solutions of the equations using data and cannot be measured in the field, they have an inherent correction for the missing terms. Thus an expansion to four terms only spreads the correction for the remaining constants over four constants (c_1, c_2, c_3 and c_4) instead of two (c_1 and c_2).

From the deviations reported above, the dual solution requirements of the boundary conditions ($\frac{Q_d}{Q_p} > 1$ and $\frac{Q_d}{Q_p} < 1$), and the apparent lack of response to

increase the accuracy upon expansion of the equation, Mr. Diskin's solution (although within engineering requirements) is not to be preferred over Prof. McPherson's solution to the pipe network problem.

2. General Pipe Network

If a general pipe network is given (Fig. 2) it can be replaced by an equivalent series pipe network by considering the equal-head contour lines. If actual pipes are connected between the points at which each contour line crosses the given network, the state of the system will not change since there will be no flow in the pipes added. If the demand is assumed now to be concentrated at the points of crossing of contours (or if the contours are plotted at such a density that a contour will pass through each pipe junction), the new system becomes in effect the series network treated previously.

For the approximate analysis given above to apply it is necessary that the general shape of the contour lines will not change as the ratio $\frac{Q_p}{Q_d}$ changes. This appears to be the case when $\frac{Q_p}{Q_d} > 1.0$ that is, when there is flow to storage but not when the flow is reversed, from storage. When the flow is reversed there are, however, certain sections of the original pipe network for which the above approximate analysis will apply.

It may be concluded that over some ranges of variation in the ratio of $\left(\frac{Q_d}{Q_p}\right)$ the head losses between two points in a network may be approximated by:

$$\frac{\Sigma h}{Q_p^m} = c_1 - c_2\left(\frac{Q_d}{Q_p}\right)$$

where c_1 and c_2 are constants depending on the pipe network.

Plots of $\frac{\Sigma h}{Q_p^{1.85}}$ versus $\frac{Q_d}{Q_p}$ (from data by MBM) indicate that separate straight lines may be fitted for the losses to any point for the two ranges of values of the ratio $\left(\frac{Q_d}{Q_p}\right)$: one line for $\left(\frac{Q_d}{Q_p}\right) > 1.0$ and one for $\left(\frac{Q_d}{Q_p}\right) < 1.0$

TABLE Ia

COMPARISON OF DISKIN'S EXPANDED AND NON-EXPANDED SOLUTIONS*

					Head Loss, Σ h, from Pump to Storage			
					Non Expanded Sol.		Expanded Sol.	
No.	Q_d	Q_p	Q_p/Q_d	Q_d/Q_p	Calc.	Diff. From Meas.	Calc.	Diff. From Meas.
1	52.5	35.01	0.667	1.50	18.16	0.16	18.16	0.16
2	52.5	35.70	0.680	1.47	19.2	0.1	18.33	0.67
3	63.0	47.25	0.750	1.33	36.46	0.36	35.27	0.83
4	52.5	39.38	0.750	1.33	26.03	0.67	25.18	1.52
5	42.0	31.50	0.750	1.33	17.22	0.38	16.66	0.34
					$c_1 = .06144$		$c_1 = .47019$	
					$c_2 = .0243$		$c_2 = 6121$	
							$c_3 = .2162$	
							$c_4 = .00398$	

*Data from A.S.C.E. Transactions Paper No. 3229, p. 1198
m = 1.85 (TORR. H.S. - P.S. to Storage)

TABLE 11

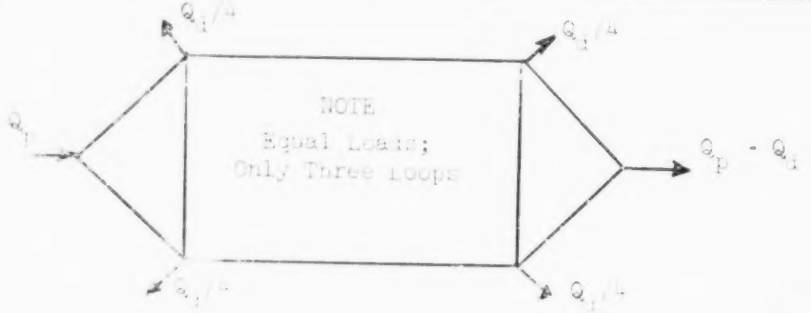
COMPARISON OF PIPE NETWORK SOLUTIONS

					Head Loss, Σh from Pumping Station to Storage						
								M H. Diskin			
No	Q_d	Q_p	Q_p/Q_d	Q_d/Q_p	Meas.	M B M	Difference	$Q_d/Q_p > 1$	Difference	$Q_d/Q_p < 1$	Difference
1	19 9	13 89	0 698	1 43	9 0	9.0	0 0	8 99	.01	$c_1 = .7153$	
2	12 3	9 31	0 757	1 32	4 0	5.0	1.0	3 99	.01	$c_2 = .6351$	
3	11 2	8 51	0 760	1 31	3 4	4.3	0.9	3 38	.02		
4	12 3	9 80	0 797	1.25	5 8	6.1	0.3	4.19	1.61		
5	10.3	8 52	0.827	1.209	4 8	5.0	0.2	3.14	1.66		
6	12.3	10 20	0 829	1 206	7.2	7.0	0.2	4 37	2.83		
7	11.2	9 30	0 830	1.205	5.1	5.9	0 8	3.69	1.41		
8	9 7	8 57	0 883	1 13	6 6	5 7	0.9	3.00	3.60		
9	10 3	9 70	0 942	1.06	8 2	8.1	0.1	3 59	4.61		
10	5 1	5 1	1 00	1 00	3 2	2 8	0.4	1 04	2.16	1.66	1.54
11	12 3	12 3	1 00	1 00	14 0	14.0	0 0	5 24	8.76	8 28	5.72
12	7 6	7 6	1 00	1 00	6 3	5 9	0 4	2.16	4.14	3 42	2 88
13	7 3	8 54	1 17	0 855	9.1	9 4	0.3	$c_1 = -.0427$		9 10	0 003
14	7 3	9 34	1 28	0 781	13 7	13 1	0.6	$c_2 = .0081$		13 69	0.01
15	5 1	8 42	1 65	0 606	20 0	17 5	2.5			17 02	2 98
16	5 1	8 92	1 73	0 578	22 0	20 7	1 3			19 95	2.05
17	5 1	9 33	1 83	0 546	25 2	25 5	0 3			22 94	2.26
18	5 1	9 69	1 90	0 526	29 0	29 6	0.6			25.47	3.53
19	7 0	13 93	1 99	0 503	60.0	62 4	2 4			51 85	8 15

Data from A S C E Transactions No 3229, p 1201
McPherson's : = 0 135 n = 3 70, m = 1 85 (BELMONT H S - P S. to Storage)

					Head Loss, Σh, from Pumping Station to Storage						
								M. H. Diskin			
No	Q_d	Q_p	Q_p/Q_d	Q_d/Q_p	Meas.	M B M.	Difference	$Q_d/Q_p > 1$	Difference	$Q_d/Q_p < 1$	Difference
TABLE III (from CE 250 Prob. 6 (5/62) McPherson's $\Phi = 1.7$, $n = 2.8$, $m = 2.0$)											
1	1.76	3.53	2.00	0.500	37.4	36.8	0.6	c_1=4.833		37.4	0.0
2	2.10	3.53	1.68	0.595	33.0	32.0	1.0	c_2=2.871		33.0	0.0
3	2.65	3.53	1.33	0.752	26.8	26.42	0.38			26.8	1.06
4	3.53	3.53	1.00	1.00	21.0	22.22	1.22	24.45	3.45	15.5	4.5
5	4.15	3.53	0.850	1.18	18.0	17.6	0.4	18.0	0.0	c_1=4.860	
6	5.27	3.53	0.670	1.49	6.9	15.5	8.6	6.9	0.0	c_2=3.716	
7	7.06	3.53	0.500	2.00	-27.0	12.2	39.2	-11.33	15.66		
TABLE IV (from CE 250 Prob. 5 (10/62) McPherson's $\Phi = 1.31$, $n = 5.2$, $m = 2.0$)											
1	2.80	4.00	1.43	0.70	66.0	65.8	0.2			66.0	0.0
2	3.20	4.00	1.25	0.80	42.9	42.2	0.2			42.9	0.0
3	2.00	3.00	1.50	0.667	42.5	42.9	0.4			41.5	1.0
4	2.40	3.00	1.25	0.80	24.1	24.1	0.0			24.15	0.05

c_1=14.44
c_2=14.233



Report on Mr. R. W. Adams' Method of Distribution Analysis

Mackenzie L. Davis

A limited numerical study of Mr. Adams' method of analysis (Tables I. II and III) revealed the following:

1. The limits of the $\frac{Q_p}{Q_d}$ ratio did not restrict design requirements. A $\frac{Q_p}{Q_d}$ range of 0.592 to 2.58 proved workable.

2. The equation ($\Sigma h = kQ_d + c$) is not limited to gravity-fed systems. <u>However, the restriction of a near-constant Q_p is essential for the equation to apply to direct pumping systems.</u> (The reason why this restriction did not apply to the gravity fed system, Table I, could not be ascertained.) (But Adams' $Q_p \cong$ Const.)

3. Determination of the unknowns from two series of data does not necessarily give the best line of fit for the variation of the system. The error incurred, although falling within the $\pm$ 5 feet stipulated by Mr. Adams, is generally greater than that from Prof. McPherson's method of solution.

No	Q_d	Q_p	Q_p/Q_d	Q_d/Q_p	Head Loss, Σh, from Pumping Station to Indicated Location: Meas	M B M	Difference	R W A.	Difference	
TABLE I (From I W E. Journal March 1963 p 139. TABLE III; Σh to Q_d location)										
1	2 23	5 75	2 58	0 388	50	50	0 0	50 0	0 0	Using:
2	3 65	6 05	1 66	0 603	55	55	0 0	54 1	-0 9	$\Sigma h = 43.55 + 2.89\ Q_d$
3	6 19	6 19	1 00	1 00	60	57 2	-2 8	61 45	+1 45	
4	9 00	6 7	0 735	1 36	65	64 7	-0 3	69 55	+4 55	
5	10 40	6 91	0 665	1 50	70	70	0 0	73 55	+3 55	
6	12 50	7 4	0 592	1 69	80	79 7	-0 3	79 65	-0 35	
TABLE II (From A S C E Transactions Paper No. 3229 p. 1198) (Σh to Storage Site)										
5	42 0	31 50	0 ,50	1 33	17 6	17 1	-0 5	17 6	0 0	Using:
12	31 5	31 5	1 00	1 00	21 2	20 3	-0 9	21 1	-0 1	$\Sigma h = 31.6 - .333\ Q_d$
15	25 1	31 38	1 25	0 80	24 8	23 5	-1 3	23 25	-1 55	($35.7 - 0.426\ Q_d$ is closer)
20	18 9	31 56	1 67	0 598	27 8	27 9	+0 1	25 3	-2 5	
TABLE III (From A S C E. Transactions Paper No 3229 p. 1198) (Σh to Storage Site)										
4	52 5	39 38	0 750	1 33	26 7	25 8	-0 9	26 7	0 0	Using:
10	39 4	39 4	1 00	1 00	31 4	30 8	-1 1	31 7	-0 2	$\Sigma h = 47.3 -$ [illegible] Q_d
14	31 5	39 38	1 25	0 80	36 9	35 2	-1 7	34 8	2 1	
19	23 6	39 41	1 67	0 598	42 0	42 1	+0 1	37 9	-4 1	

(Note: Q_p = Const for above)

APPENDIX III

TABLES LISTING RESULTS FOR ALL COMPUTER RUNS

36-U1-U1-S1

Φ = 0.02975

N = 2.369

Number of Branches Between The Pump Input Node and the Storage Outlet Node = 12 (ΣB=12)

Q_p/Q_i	$\Sigma h/KQ_i^2$	$\Phi(Q_p/Q_i)^N$	Diff
0.500	0.0000	--	--
0.600	0.0080	0.0089	0.0009
0.750	0.0150	0.0150	0.0000
1.000	0.0306	0.0297	-0.0009
1.500	0.0799	0.0778	-0.0021
2.000	0.1623	0.1537	-0.0086
3.000	0.4254	0.4015	-0.0239
4.000	0.8203	0.7936	-0.0267
5.000	1.5464	1.5464	0.0000
10.000	5.9521	6.9542	1.0021

36-U1-U1-S2

Φ = 0.03653

N = 2.175

ΣB = 10

Q_p/Q_i	$\Sigma h/KQ_i^2$	$\Phi(Q_p/Q_i)^N$	Diff
0.400	0.0016	0.0050	0.0033
0.500	0.0061	0.0080	0.0019
0.600	0.0111	0.0120	0.0009
0.750	0.0196	0.0196	0.0000
1.000	0.0368	0.0365	-0.0002
1.500	0.0895	0.0880	-0.0015
2.000	0.1691	0.1650	-0.0040
3.000	0.4087	0.3985	-0.0102
4.000	0.7559	0.7450	-0.0109
5.000	1.2104	1.2104	0.0000
10.000	5.0949	5.4663	0.3714

36-U1-U1-S3

Φ = 0.04528

N = 2.132

ΣB = 8

Q_p/Q_i	$\Sigma h/KQ_i^2$	$\Phi(Q_p/Q_i)^N$	Diff.
0.400	0.0055	0.0065	0.0010
0.500	0.0086	0.0105	0.0019
0.600	0.0143	0.0152	0.0010
0.750	0.0245	0.0245	0.0000
1.000	0.0458	0.0452	-0.0006
1.500	0.1092	0.1075	-0.0017
2.000	0.2028	0.1985	-0.0043
3.000	0.4809	0.4713	-0.0096
4.000	0.8804	0.8705	-0.0100
5.000	1.4010	1.4010	0.0000
10.000	5.8519	6.1409	0.2890

36-U1-U1-84

Φ = 0.05942

N = 2.111

ΣB = 6

Q_p/Q_d	$\Sigma h/kQ_d^2$	$\Phi(Q_p/Q_d)^N$	Diff.
0.400	0.0051	0.0086	0.0035
0.500	0.0117	0.0157	0.0021
0.600	0.0192	0.0202	0.0010
0.750	0.0323	0.0323	0.0000
1.000	0.0602	0.0505	-0.0007
1.500	0.1419	0.1398	-0.0021
2.000	0.2612	0.2566	-0.0045
3.000	0.6137	0.6039	-0.0099
4.000	1.1183	1.1083	-0.0100
5.000	1.7750	1.7750	0.0000
10.000	7.3379	7.6652	0.3273

36-N5-U1-S1

Φ = 0.03631

N = 2.387

ΣB = 12

Q_p/Q_d	$\Sigma h/kQ_d^2$	$\Phi(Q_p/Q_d)^N$	Diff.
0.500	0.0000	--	--
0.600	0.0073	0.0107	0.0035
0.750	0.0183	0.0183	0.0000
1.000	0.0375	0.3630	-0.0012
1.500	0.0995	0.0955	-0.0039
2.000	0.2050	0.1899	-0.0151
3.000	0.5340	0.4999	-0.0340
4.000	1.0306	0.9934	-0.0371
5.000	1.6823	1.6823	0.0000
10.000	7.1846	8.8519	1.6673

36-N25-U1-S1

Φ = 0.04186

N = 2.398

ΣB = 12

Q_p/Q_d	$\Sigma h/kQ_d^2$	$\Phi(Q_p/Q_d)^N$	Diff.
0.500	0.0000	---	--
0.600	0.0083	0.0123	0.0040
0.750	0.0210	0.0210	0.0000
1.000	0.0433	0.0419	-0.0014
1.500	0.1162	0.1107	-0.0055
2.000	0.2377	0.2207	-0.0170
3.000	0.6284	0.5855	-0.0429
4.000	1.2096	1.1633	-0.0463
5.000	1.9867	1.9867	0.0000
10.000	8.7990	10.4780	1.6790

36-M5-U1-S1

Φ = 0.03837

N = 2.397

ΣB = 12

Q_p/Q_d	$\Sigma h/KQ_d^2$	$\Phi(Q_p/Q_d)^N$	Diff.
0.500	0.0000	--	--
0.600	0.0076	0.0113	0.0037
0.750	0.0193	0.0193	0.0000
1.000	0.0397	0.0383	-0.0015
1.500	0.1064	0.1014	-0.0049
2.000	0.2175	0.2021	-0.0154
3.000	0.5729	0.5341	-0.0388
4.000	1.1063	1.0643	-0.0420
5.000	1.8170	1.8170	0.0000
10.000	8.0381	9.5695	1.5314

36-M25-U1-S1

Φ= 0.04748

N = 2.418

ΣB = 12

Q_p/Q_d	$\Sigma h/KQ_d^2$	$\Phi(Q_p/Q_d)^N$	Diff
0.500	0.0000	--	--
0.600	0.0094	0.0138	0.0044
0.750	0.0237	0.0237	0.0000
1.000	0.0493	0.0475	-0.0018
1.500	0.1347	0.1265	-0.0081
2.000	0.2770	0.2537	-0.0233
3.000	0.7321	0.6762	-0.0559
4.000	1.4151	1.3556	-0.0595
5.000	2.3251	2.3251	0.0000
10.000	10.2910	12.4234	2.1324

36-025*-U1-S1

Φ= 0.04757

N = 2.727

ΣB = 12

Q_p/Q_d	$\Sigma h/KQ_d^2$	$\Phi(Q_p/Q_d)^N$	Diff
0.500	--	--	--
0.600	--	--	--
0.750	0.0217	0.0217	0.0000
1.000	0.0552	0.0475	-0.0076
1.500	0.1727	0.1436	-0.0291
2.000	0.3917	0.3148	-0.0770
3.000	1.1388	0.9609	0.1619
4.000	2.2792	2.0856	0.1936
5.000	3.8837	3.8837	0.0000
10.000	17.6516	25.5408	7.8892

*(Same as 36-025-U1-S1, except that a 25K branch adjacent to S1 was inadvertently set at a value of K).

36-05-U1-S1

Φ = 0.03944

N = 2.411

ΣB = 12

Q_p/Q_i	$\Sigma h/KQ_i^2$	$\Phi(Q_p/Q_i)^N$	Diff.
0.500	0.0000	--	--
0.600	0.0078	0.0115	0.0037
0.750	0.0197	0.0197	0.0000
1.000	0.0410	0.0394	-0.0016
1.500	0.1110	0.1048	-0.0062
2.000	0.2279	0.2097	-0.0182
3.000	0.6016	0.5573	-0.0443
4.000	1.1625	1.1152	-0.0473
5.000	1.9097	1.9071	0.0000
10.000	8.4506	10.1544	1.7037

36-L1-U1-S1

Φ = 0.03962

N = 2.433

ΣB = 12

Q_p/Q_i	$\Sigma h/KQ_i^2$	$\Phi(Q_p/Q_i)^N$	Diff.
0.500	0.0000	--	--
0.600	0.0078	0.0114	0.0037
0.750	0.0197	0.0197	0.0000
1.000	0.0415	0.0396	-0.0018
1.500	0.1145	0.1063	-0.0082
2.000	0.2362	0.2139	-0.0222
3.000	0.6254	0.5737	-0.0518
4.000	1.2095	1.1550	-0.0544
5.000	1.9877	1.9877	0.0000
10.000	8.8000	10.7518	1.9518

36-U1-L1-S1

Φ = 0.03171

N = 2.340

ΣB = 12

Q_p/Q_i	$\Sigma h/KQ_i^2$	$\Phi(Q_p/Q_i)^N$	Diff.
0.500	0.0014	0.0063	0.0049
0.600	0.0072	0.0096	0.0024
0.750	0.0162	0.0162	0.0000
1.000	0.0526	0.0517	-0.0009
1.500	0.0845	0.0818	-0.0026
2.000	0.1693	0.1605	-0.0088
3.000	0.4578	0.4144	-0.0434
4.000	0.8379	0.8124	-0.0255
5.000	1.3693	1.3693	0.0000
10.000	6.0014	6.9518	0.9504

36-L1-L1-S1

Φ = 0.03171

N = 2.340

ΣB = 12

Q_p/Q_d	$\Sigma h/KQ_d^2$	$\Phi(Q_p/Q_d)^N$	Diff.
0.500	0.0054	0.0089	0.0055
0.600	0.0109	0.0137	0.0027
0.750	0.0252	0.0252	0.0000
1.000	0.0480	0.0458	-0.0023
1.500	0.1288	0.1193	-0.0095
2.000	0.2583	0.2356	-0.0226
3.000	0.6631	0.6145	-0.0486
4.000	1.2628	1.2134	-0.0494
5.000	2.0567	2.0567	-0.0000
10.000	8.9473	10.5925	-1.6452

36-U1-U1-S6

Φ = 0.01108

N = 3.565

ΣB = 14

Q_p/Q_d	$\Sigma h/KQ_d^2$	$\Phi(Q_p/Q_d)^N$	Diff.
0.750	0.0040	0.0040	0.0000
1.000	0.0263	0.0111	0.0152
1.500	0.1042	0.0470	0.0573
2.000	0.2820	0.1311	0.1508
3.000	0.9361	0.5566	0.3794
4.000	1.9889	1.5524	0.4364
5.000	3.4397	3.4397	0.0000
10.000	16.6731	40.7198	-24.0468

25-U1-U1-S1

Φ = 0.03447

N = 2.373

ΣB = 10

Q_p/Q_d	$\Sigma h/KQ_d^2$	$\Phi Q_p/Q_d^N$	Diff
0.500	0.0000	--	--
0.600	0.0069	0.0104	0.0035
0.750	0.0174	0.0174	0.0000
1.000	0.0355	0.0345	-0.0010
1.500	0.0931	0.0902	-0.0029
2.000	0.1885	0.1796	-0.0089
3.000	0.4964	0.4675	-0.0289
4.000	--	--	--
5.000	1.5710	1.5710	0.0000
10.000	6.9514	8.1428	1.1914

25-U1-U1-S2

Φ = 0.04405

N = 2.174

ΣB = 8

Q_p/Q_d	$\Sigma h/KQ_d^2$	$\Phi(Q_p/Q_d)^N$	Diff
0.500	--	--	--
0.600	0.0134	0.0145	0.0010
0.750	0.0235	0.0235	0.0000
1.000	0.0444	0.0441	-0.0003
1.500	0.1082	0.1064	-0.0018
2.000	0.2039	0.1987	-0.0050
3.000	0.4927	0.4799	-0.0129
4.000	0.9104	0.8968	-0.0136
5.000	1.4566	1.4566	0.0000
10.000	6.1226	6.5726	0.4500

25-U1-U1-S3

Φ = 0.05798

N = 2.128

ΣB = 6

Q_p/Q_d	$\Sigma h/KQ_d^2$	$\Phi(Q_p/Q_d)^N$	Diff
0.500	0.0110	0.0155	0.0045
0.600	0.0184	0.0195	0.0011
0.750	0.0315	0.0315	0.0000
1.000	0.0586	0.0579	-0.0007
1.500	0.1373	0.1374	+0.0001
2.000	0.2589	0.2534	-0.0054
3.000	0.6126	0.6006	-0.0121
4.000	1.1139	1.1076	-0.0063
5.000	1.7808	1.7808	0.0000
10.000	7.5892	7.7830	0.1938

16-U1-U1-S1

Φ = 0.04108

N = 2.379

ΣB = 8

Q_p/Q_d	$\Sigma h/KQ_d^2$	$\Phi(Q_p/Q_d)^N$	Diff
0.500	0.0000	--	--
0.600	0.0085	0.0121	0.0036
0.750	0.0208	0.0207	0.0000
1.000	0.0404	0.0411	+0.0007
1.500	0.1117	0.1077	-0.0039
2.000	0.2270	0.2137	-0.0133
3.000	0.5904	0.5904	-0.0000
4.000	1.1504	1.1198	-0.0306
5.000	1.8887	1.8887	0.0000
10.000	8.3940	9.8614	1.4674

16-U1-U1-S2

Φ = 0.05592

N = 2.170

ΣB = 6

Q_p/Q_d	$\Sigma h/KQ_d^2$	$\Phi(Q_p/Q_d)^N$	Diff.
0.400	0.0035	0.0076	0.0044
0.500	0.0099	0.0124	0.0025
0.600	0.0172	0.0185	0.0013
0.750	0.0299	0.0299	0.0000
1.000	0.0564	0.0559	-0.0004
1.500	0.1376	0.1347	-0.0028
2.000	0.2589	0.2516	-0.0073
3.000	0.6252	0.6084	-0.0168
4.000	1.1490	1.1318	-0.0172
5.000	1.8366	1.8366	0.0000
10.000	7.7013	8.2629	0.5610

16-U1-U1-S5

Φ = 0.07790

N = 2.390

ΣB = 4

Q_p/Q_d	$\Sigma h/KQ_d^2$	$\Phi(Q_p/Q_d)^N$	Diff.
0.600	0.0156	0.0229	0.0076
0.750	0.0592	0.0592	0.0000
1.000	0.0855	0.0780	-0.0057
1.500	0.2178	0.2053	-0.0125
2.000	0.3951	0.4085	-0.0133
3.000	1.1566	1.0797	-0.0769
4.000	2.2239	2.1417	-0.0822
5.000	3.6511	3.6511	-0.0000
10.000	16.1508	19.1458	3.0042

16-M5-U1-S1

Φ = 0.06470

N = 2.403

ΣB = 8

Q_p/Q_i	$\Sigma h/KQ_i^2$	$\Phi(Q_p/Q_i)^N$	Diff
0.500	0.0000	--	--
0.600	0.0129	0.0189	0.0061
0.750	0.0505	0.0505	0.0000
1.000	0.0971	0.0947	-0.0024
1.500	0.1806	0.1714	-0.0092
2.000	0.3068	0.2994	-0.0075
3.000	0.9759	0.9330	-0.0429
4.000	1.8858	1.8099	-0.0759
5.000	3.0940	3.0940	0.0000
10.000	13.0409	14.8158	1.7751

16-M25-U1-S1

Φ = 0.05503

N = 2.418

ΣB = 8

Q_p/Q_i	$\Sigma h/BQ_i^2$	$\Phi(Q_p/Q_i)^N$	Diff
0.500	0.0000	--	--
0.600	0.0108	0.0160	0.0052
0.750	0.0274	0.0274	0.0000
1.000	0.0573	0.0551	0.0022
1.500	0.1562	0.1467	0.0095
2.000	0.3211	0.2941	0.0270
3.000	0.8489	0.7859	0.0630
4.000	1.6408	1.5717	0.0691
5.000	2.6959	2.6959	0.0000
10.000	11.9508	14.4065	2.4757

16-05-U1-S1

Φ = 0.05089

N = 2.425

ΣB = 8

Q_p/Q_i	$\Sigma h/BQ_i^2$	$\Phi(Q_p/Q_i)^N$	Diff
0.500	0.0000	--	--
0.600	0.0100	0.0147	0.0047
0.750	0.0253	0.0253	0.0000
1.000	0.0532	0.0508	-0.0025
1.500	0.1458	0.1360	-0.0098
2.000	0.3000	0.2733	-0.0267
3.000	0.7946	0.7304	-0.0641
4.000	1.5341	1.4672	-0.0669
5.000	2.5209	2.5209	0.0000
10.000	11.1597	13.4368	2.2771

16-025-U1-S1

Φ = 0.06267

N = 2.468

ΣB = 8

Q_p/Q_i	$\Sigma h/BQ_i^2$	$\Phi(Q_p/Q_i)^N$	Diff
0.500	0.0000	--	--
0.600	0.0130	0.0179	0.0049
0.750	0.0309	0.0309	0.0000
1.000	0.0647	0.0627	-0.0020
1.500	0.1764	0.1709	-0.0055
2.000	0.3617	0.3441	-0.0176
3.000	0.9600	0.9192	-0.0408
4.000	1.8500	1.8170	-0.0330
5.000	3.0317	3.0317	0.0000
10.000	13.4542	16.8464	3.3922

16-P1-U1-85

Φ = 0.09820

N = 2.846

ΣB = 4

Q_p/Q_d	$\Sigma h/KQ_d^2$	$\Phi(Q_p/Q_d)^N$	Diff.
0.500	--	--	--
0.600	--	--	--
0.750	0.0432	0.0432	0.0000
1.000	0.1371	0.0981	-0.0390
1.500	0.4449	0.3114	-0.1335
2.000	0.9864	0.7059	-0.2805
3.000	2.8335	2.2386	-0.5951
4.000	5.6983	5.0761	-0.6222
5.000	9.5214	9.5792	0.0000
10.000	44.2119	68.8727	+24.6606

16-P1-P1-85

Φ = 0.1426

N = 3.628

ΣB = 4

Q_p/Q_i	$\Sigma h/KQ_i^2$	$\Phi(Q_p/Q_i)^N$	Diff.
0.600	0.0051	0.0573	0.0520
0.750	0.0669	0.0669	0.0000
1.000	0.1680	0.1426	-0.0254
1.500	0.4871	0.4138	-0.0733
2.000	1.0549	0.8814	-0.1735
3.000	2.9536	2.5585	-0.3951
4.000	5.8672	5.4492	-0.4180
5.000	9.7926	9.7926	0.0000
10.000	44.6665	60.5599	15.8934

16-P1-S1-85

Φ = 0.11940

N = 3.732

ΣB = 4

Q_i/Q_i	$\Sigma h/KQ_i^2$	$\Phi(Q_p/Q_i)^N$	Diff.
0.750	0.0545	0.0545	0.0000
1.000	0.1617	0.1195	-0.0422
1.500	0.4651	0.3514	-0.1137
2.000	1.0193	0.7952	-0.2241
3.000	2.8955	2.4015	-0.4940
4.000	5.7877	5.2699	-0.5178
5.000	9.6953	9.6953	0.0000
10.000	44.4884	59.4434	14.9550

9-U1-U1-S1

Φ = 0.05108

N = 2.384

ΣB = 6

Q_p/Q_d	$\Sigma h/KQ_d^2$	$\Phi(Q_p/Q_d)^N$	Diff.
0.500	0.0000	--	--
0.600	0.0103	0.0151	0.0048
0.750	0.0257	0.0257	0.0000
1.000	0.0530	0.0511	-0.0018
1.500	0.1398	0.1343	-0.0055
2.000	0.2846	0.2667	-0.0179
3.000	0.7479	0.7011	-0.0469
4.000	1.4428	1.3919	-0.0508
5.000	2.3695	2.3695	0.0000
10.000	10.4774	12.3691	1.8918

9-U1-L1-S1

Φ = 0.05560

N = 2.343

ΣB = 6

Q_p/Q_i	$\Sigma h/KQ_i^2$	$\Phi(Q_p/Q_i)^N$	Diff.
0.500	0.0024	0.0110	0.0086
0.600	0.0126	0.0168	0.0042
0.750	0.0284	0.0283	0.0000
1.000	0.0578	0.0556	-0.0021
1.500	0.1496	0.1438	-0.0058
2.000	0.2993	0.2821	-0.0172
3.000	0.7726	0.7294	-0.0432
4.000	1.4775	1.4312	-0.0463
5.000	2.4141	2.4141	0.0000
10.000	10.5715	12.2475	1.6760

9-L1-L1-S1

Φ = 0.09369

N = 2.369

ΣB = 6

Q_p/Q_i	$\Sigma h/KQ_i^2$	$\Phi(Q_p/Q_i)^N$	Diff.
0.500	0.0081	0.0181	0.0102
0.600	0.0227	0.0280	0.0051
0.750	0.0475	0.0475	0.0000
1.000	0.0993	0.0936	-0.0056
1.500	0.2675	0.2448	-0.0227
2.000	0.5356	0.4837	-0.0517
3.000	1.3709	1.2640	-0.1068
4.000	2.6051	2.4984	-0.1067
5.000	4.2384	4.2384	0.0000
10.000	18.9895	21.8873	2.8978

2-L2-L1-S1

Φ = 0.20205

N = 2.400

TB = 6

Q_p/Q_d	$\Sigma n/KQ_d^E$	$\Phi(Q_p/Q_d)^N$	Diff.
0.400	0.0060	0.0225	0.0165
0.500	0.0225	0.0385	0.0160
0.600	0.0500	0.0593	0.0093
0.750	0.1271	0.1271	0.0000
1.000	0.2613	0.2819	-0.0194
1.500	0.6181	0.6545	-0.0365
2.000	1.2367	1.0661	-0.1705
3.000	3.1598	2.8206	-0.3391
4.000	5.9308	5.6254	-0.3053
5.000	9.6050	9.6050	0.0000
10.000	41.3235	50.7234	9.3999

4-U1-U1-S1

Φ = 0.06901

N = 2.383

TB = 4

Q_p/Q_d	$\Sigma n/KQ_d^E$	$\Phi(Q_p/Q_d)^N$	Diff.
0.500	0.0000	--	--
0.600	0.0138	0.0205	0.0068
0.750	0.0547	0.0547	0.0000
1.000	0.0720	0.0691	-0.0030
1.500	0.1098	0.1814	-0.0076
2.000	0.3643	0.3600	-0.0043
3.000	1.0095	0.9464	-0.0631
4.000	1.9468	1.8782	-0.0686
5.000	3.1968	3.1968	0.0000
10.000	14.1347	16.6780	2.5433